EUCLID's ELEMENTS.

The Whole Power

FIFTEEN BOOKS

Compendiously Demonstrated

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And Translated out of the Latin.

Καθαρμοὶ ψυχῆς λογικῆς εἰσιν αἱ μαθηματικαὶ ἐπιστήμαι.
Hierocl.

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The Author's
PREFACE.

IN order to the Reader's Satisfaction concerning the Book put into his Hand, I am to advertiſe him of ſome few things, and that according to the nature of the Work, briefly; as followeth. My Undertaking aimed principally at two Ends. The firſt of which was to conjoyn the greateſt Compendiouſneſs of Demonſtration with as much Perſpicuity as the quality of the Subject would admit; that ſo the Volume might bear no bigger Bulk than would render it conveniently portable. Which

I have ſo far attained, that though poſſibly ſome other Perſon might with greater Curioſity, yet (I preſume) none could with more Conciſeneſs have demonſtrated moſt Propoſitions; eſpecially ſince I have altered nothing in the number and order of the Propoſitions, nor taken the liberty to leave out any one of *Euclid*'s as leſs neceſſary, or to reduce certain of the eaſieſt into the Claſſis of *Axioms*. Which notwithſtanding ſome have done; as that moſt accurate Geometrician *Andr. Tacquet*, whom I mention the rather, becauſe I eſteem it ingenuous to acknowledge ſome things taken from him. And, indeed, I ſhould have attempted nothing after his moſt elegant Edition, had it not pleaſed that learned Perſon to publiſh only Eight of *Euclid*'s Books illuſtrated by his Pains, either ſlighting or undervaluing the other Seven as leſs relating to the Elements of Geometry. But I had a different Purpoſe from the Beginning; not to compoſe Elements of Geometry any-wiſe at my Diſ-

Diſcretion, but to demonſtrate *Euclid* himſelf, and all of him, and that with all poſſible Brevity. For as for Four of his Books, the Seventh, Eighth, Ninth, and Tenth, although they do not ſo nearly pertain to the Elements of plane and ſolid Geometry, as the ſix firſt and the two ſubſequent; yet no man that has arriv'd to any meaſure of skill in Geometry, is ignorant how exceedingly uſeful they are in Geometrical matters, as well in regard of the very near Alliance between Arithmetick and Geometry, as for the Knowledge of Commenſurable and Incommenſurable Magnitudes which is highly important to the underſtanding both of plane and ſolid Figures. And the noble Theory of the five regular Bodies, contained in the three laſt Books, could not be omitted without Prejudice and Injury; ſince our Author of theſe Elements, being a Sectator of *Plato*'s School, is reported to have compiled the whole Syſteme only in reference to that Contemplation; which

Proclus atteſteth in theſe Words; Ὅθεν δὴ καὶ τῆς Συμπάσης στοιχειώσεως τέλος προεστήσατο τὴν τῶν καλουμένων Πλατωνικῶν σημάτων Σύστασιν. Moreover, I was eaſily induc'd to believe, that it would be acceptable to all Lovers of theſe Sciences to have the entire work of *Euclid* by them, as it is uſually cited and recommended by all men. Wherefore I determin'd to leave out no Book or Propoſition of thoſe which are found in *Peter Herigon*, whoſe Footſteps I became neceſſitated to follow cloſely, by having reſolved to make uſe for the moſt part of the Schemes of his Book, upon a foreſight that my ſpeedy Departure out of *England*, would not allow me time to deſcribe New, although I ſometimes deſired ſo to do. Upon the ſame Account alſo I purpoſed to uſe generally no other than *Euclid*'s own Demonſtrations, contracted into a more ſuccinct form, ſaving perchance in the ſecond and thirteenth, and ſparingly in the ſeventh, eighth, and ninth Books, where it ſeem'd convenient to vary ſomething

from

from him. So that it may be reaſonably hoped, that in this Particular our own Deſign, and the Wiſhes of the Studious, are in ſome manner ſatisfied.

The other End aimed at, was in favour of their deſires who more affect Symbolical than Verbal Demonſtrations. In which kind, ſeeing moſt of our own Nation are accuſtomed to the Notes of Mr. *Oughtred*, I eſteemed it more convenient to make uſe of them principally throughout. For no man hitherto, that I know of, ſaving only *Peter Herigon*, has attempted to ſet forth and interpret *Euclid* according to this way. The Method of which moſt learned Perſon, though in many other reſpects very excellent, and exactly accommodated to his peculiar purpoſe, ſeem'd to me notwithſtanding doubly defective. Firſt, in that, whereas of ſeveral Propoſitions brought to the proving of ſome one Theorem or Problem, the latter does not always depend on the former; yet when they do cohere one with another, and when

when not, cannot readily enough be known, either from their Order or any other way; whence it not ſeldom comes to paſs, that through the want of Conjunctions and Adjectives, *ergo*, *rurſus*, &c. there ariſes difficulty and occaſion of doubting, eſpecially to ſuch as are but little vers'd therein. And in the next place, it oftentimes falls out that the ſaid Method cannot avoid ſuperfluous Repetitions, whereby the Demonſtrations become ſometimes prolix, and ſometimes perplex'd and intricate. All which Inconveniences are eaſily remedied in our Way by the intermingling of Words and Signs at Diſcretion. And thus much may ſuffice to be premiſed concerning the Intent and Method of this Compendium. I ſhall not alledge in favour of my ſelf the ſcantneſs of time allotted to this Work, nor the Avocations of Affairs, nor the ſcarcity of Helps to this ſort of Studies amongſt us (as I might not untruly) out of fear leſt my Performance ſhould not throughly pleaſe every Body:

But

But I wholly ſubmit to the fair Cenſure and Judgment of the Ingenuous Reader, what I have undertaken for the advantage of his Studies; to be approved, if he find it ſerviceable thereunto; or, if otherwiſe, rejected.

Ad amiciſſimum Virum, *J. C.* de *EUCLIDE* contracto, Εὐρημισμός.

FActum bene! didicit Laconice loqui
Senex profundus, & aphoriſmos induit.
Immenſa dudum margo commentarii
Diagramma circuit minutum; utque Inſula
Problema breve natabat in vaſto mari.
Sed unda jam detumuit; & gloſſa arctior
Stringit Theoremata: minoris anguli
Lateribus ecce totus Euclides *jacet,*
Incluſus olim velut Homerus in nuce;
Pluteoque ſarcina modo qui incubuit, levis
En ſit manipulus. Pelle in exigua latet
Ingens Matheſis, matris utero Hercules,
In glande quercus, vel Ithaca Eurus in pila.
Nec mole dum decreſcit, uſu fit minor;
Quin auctior jam evadit, & cumulatius
Contracta prodeſt erudita pagina.
Sic ubere magis liquor è preſſo effluit;
Sic pleniori vaſa inundat ſanguinis
Torrente cordis Syſtole; ſic fuſius

Pro-

Procurrit æquor ex Abyla angustiis.
Tantilli operis ars tanta referenda unice est
BAROVIANO *nomini, ac solertia.*
Sublimis euge mentis ingenium potens!
Cui invium nil, arduum esse nil solet;
Sic usque pergas prospero conamine,
Radiusque multum debeat ac abacus tibi;
Sic crescat indies feracior seges,
Simili colonum germine assiduo beans.
Specimen futuræ messis hic siet labor,
Magnæque famæ illustria hæc præludia.
Juvenis dedit qui tanta, quid dabit senex?

Car. Robotham, *Cantab.*
Coll. Trin. Sen. Soc.

The

The Explication of the Signs or Characters.

=	Signifies	Equal.
⊏		Greater.
⊐		Leſſer.
+		More, or to be added.
−		Leſs, or to be ſubſtracted.
−:		The Difference, or Exceſs; alſo, that all the quantities which follow, are to be ſubtracted, the Signs not being changed.
×		Multiplication, or the drawing one ſide of a Rectangle into another. The ſame is denoted by the Conjunction of Letters; as AB=A×B.
√		The Side or Root of a Square, or Cube, &c.
Q & q		A Square.
C & c		A Cube.
Q. Q.		The ratio of a ſquare number to a ſquare number.

Other Abbreviations of Words, wherever they occur, the Reader will, without trouble, underſtand of himſelf; ſaving ſome few, which, being of leſs general Uſe, we refer to be explained in their own places.

THE

THE FIRST BOOK OF Euclid's *Elements.*

Definitions.

I. A Point is that which hath no part.

II. A line is a longitude without latitude.

III. The ends or limits of a line are points.

IV. A right line is that which lies equally betwixt its points.

V. A ſuperficies is that which hath onely longitude and latitude.

VI. The extremes or limits of a ſuperficies are lines.

VII. A plain ſuperficies is that which lies equally betwixt its lines

VIII. A plain angle is the inclination of two lines the one to the other, the one touching the other in the ſame plain, yet not lying in the ſame ſtrait line.

IX. And if the lines which contain the angle be right lines, it is called a right-lined angle.

X. When a right line CG, ſtanding upon a right line AB, makes the angles on either ſide thereof CGA, CGB, equal one to the other, then both theſe equal angles are right angles; and the right line CG, which ſtandeth on the other, is termed a perpendicular to that AB, whereon it ſtandeth.

Note. *When ſeveral angles meet at the ſame point (as at G) each particular angle is deſcribed by three letters; whereof the middle letter ſheweth the angular point, and the two other letters the lines that make that angle. As the angle which the right lines CG, AG make at G, is called CGA or AGC.*

XI. An obtuſe angle is that which is greater than a right angle, as ACB.

XII. An acute angle is that which is leſs than a right angle, as ACD.

XIII. A limit or term is the extremity of any thing.

XIV. A figure is that which is contained under one or more terms.

XV. A circle is a plain figure contained under one line, which is called a circumference; unto which all lines drawn from one point within the figure, and falling upon the circumference thereof, are equal the one to the other.

XVI. And that point is called the center of the circle.

XVII. A diameter of a circle is a right line drawn through the center thereof, and ending at the circumference on either ſide, dividing the circle into two equal parts.

XVIII. A ſemicircle is a figure which is contained under the diameter, and under that part of the circumference which is cut off by the diameter.

In

In the circle EABCD, E is the center, AC the diameter, ABC the ſemicircle.

XIX. Right-lined figures are ſuch as are contained under right lines.

XX. Three-ſided or trilateral figures are ſuch as are contained under three right lines.

XXI. Four-ſided or quadrilateral figures are ſuch as are contained under four right lines.

XXII. Many-ſided figures are ſuch as are contained under more right lines than four.

XXIII. Of trilateral figures, that is an equilateral triangle, which hath three equal ſides, as the triangle A.

XXIV. Iſoſceles is a triangle which hath onely two ſides equal, as the triangle B.

XXV. Scalenum is a triangle, whoſe three ſides are all unequal, as C.

XXVI. Of theſe trilateral figures, a right-angled triangle is that which hath one right angle, as the triangle D, fig. 26. def.

XXVII. An amblygonium, or obtuſe-angled triangle, is that which hath one angle obtuſe, as C, fig. 25. def.

XXVIII. An oxygonium, or acute-angled triangle, is that which hath three acute angles, as A, fig. 23. def.

An equiangular, or equal-angled figure, is that whereof all the angles are equal. Two figures are equiangular, if the ſeveral angles of the one figure be equal to the ſeveral angles of the other. The ſame is to be underſtood of equilateral figures.

XXIX. Of quadrilateral, or four-ſided figures, a ſquare is that whoſe ſides are equal, and angles right, as ABCD.

XXX. A figure on the one part longer, or a long ſquare, is that which hath right angles, but not equal ſides, as ABCD.

XXXI.

XXXI. A rhombus, or diamond figure, is that which hath four equal sides, but is not right-angled, as E.

XXXII. A rhomboides, or diamond-like figure, is that whose opposite sides and opposite angles are equal; but hath neither equal sides nor right angles, as GLMH.

XXXIII. All other quadrilateral figures besides these are called trapezia or tables, as GNDH.

XXXIV. Parallel or equidistant right lines are such, which being in the same plain superficies, if infinitely produced, would never meet, as A and B.

A———
B———

XXXV. A parallelogram is a quadrilateral figure, whose opposite sides are parallel or equidistant, as GLHM.

XXXVI. In a parallelogram ABCD, when a diameter AC, and two lines EF, HI parallel to the sides, cutting the diameter in one and the same point G, are drawn so that the parallelogram be divided by them into four parallelograms; those two DG, GB, through which the diameter passeth not, are called complements; and the other two HE, FI, through which the diameter passeth, the parallelograms standing about the diameter.

A Problem is, when something is proposed to be done or effected.

A Theorem is, when something is proposed to be demonstrated.

A Corollary is, a consectary, or some consequent truth gained from a precedent demonstration.

A Lemma is, the demonſtration of ſome premiſe, whereby the proof of the thing in hand becomes the ſhorter.

Poſtulates or Petitions.

1. FRom any point to any point to draw a right line.

2. To produce a right line finite ſtreight forth continually.

3. With any center and diſtance to deſcribe a circle.

Axioms.

1. THings equal to the ſame third, are alſo equal one to the other.

As A=B=C: therefore A=C: or therefore all A,B,C are equal the one to the other.

Note. *When ſeveral quantities are joyned the one to the other continually with this mark =; the firſt quantity is by virtue of this axiom equal to the laſt, and every one to every one. In which caſe we often abſtain from citing the axiom, for brevity's ſake, although the force of the conſequence depend thereon.*

2. If to equal things you add equal things, the wholes ſhall be equal.

3. If from equal things you take away equal things, the things remaining will be equal.

4. If to unequal things you add equal things, the wholes will be unequal.

5. If from unequal things you take away equal things, the remainders will be unequal.

6. Things which are double to the ſame third, or to equal things, are equal one to the other. Underſtand the ſame of triple, quadruple, &c.

7. Things which are half of one and the ſame thing, or of things equal, are equal the one to the other. Conceive the ſame of ſubtriple, ſubquadruple, &c.

8. Things which agree together are equal one to the other.

The converſe of this axiom is true in right lines and angles, but not in figures, unleſs they be alike.

Moreover, magnitudes are ſaid to agree, when the parts of the one being applied to the parts of the other, they fill up an equal or the ſame place.

9. Every whole is greater than its part.

10. Two right lines cannot have one and the ſame ſegment (or part) common to them both.

11. Two right lines meeting in the ſame point, if they be both produced, they ſhall neceſſarily cut one the other in that point.

12. All right angles are equal the one to the other.

13. If a right line BA falling on two right lines AD, CB, make the internal angles on the ſame ſide BAD + ABC leſs than two right angles, thoſe two right lines produced ſhall meet on that ſide, where the angles are leſs than two right angles.

14. Two right lines do not contain a ſpace.

15. If to equal things you add things unequal, the exceſs or difference of the wholes ſhall be equal to the exceſs of the lines added.

16. If to unequal things equal things be added, the exceſs of the wholes ſhall be equal to the exceſs of thoſe which were at firſt.

17. If from equal things unequal things be taken away, the exceſs of the remainders ſhall be equal to the exceſs of the things taken away.

18. If

18. If from things unequal things equal be taken away, the exceſs of the remainders ſhall be equal to the exceſs of the wholes.

19. Every whole is equal to all its parts taken together.

20. If one whole be double to another, and that which is taken away from the firſt, to that which is taken away from the ſecond, the remainder of the firſt ſhall be double to the remainder of the ſecond.

The Citations are to be underſtood in this manner: when you meet with two numbers, the firſt ſhews the Propoſition, the ſecond the Book; as by 4. 1. you are to underſtand the Fourth Propoſition of the Firſt Book: and ſo of the reſt. Moreover ax. denotes Axiom, poſt. Poſtulate, def. Definition, ſch. Scholium, cor. Corollary.

PROPOSITION I.

UPON a finite right line given AB, to describe an equilateral triangle ACB.

With the centers A and B, and common distance AB or BA, [a] describe two circles cutting each other in the point C; from whence [b] draw two right lines CA, CB: then is AC [c] = AB [c] = BC [d] = AC. [e] Wherefore the triangle ACB is equilateral. Which was to be done.

[a] 3 *post.*
[b] 1 *post.*
[c] 15 *def.*
[d] 1 *ax.*
[e] 23 *def.*

Scholium.

After the same manner upon the line AB may be described an Isosceles triangle, if the distances of the equal circles be taken greater or less than the line AB.

PROP. II.

From a point given A, to draw a right line AG equal to a right line given BC.

With the center C, and distance CB, [a] describe the circle CBE. [b] Joyn AC, upon which raise the equilateral triangle ADC. [d] Produce DC to E. With the center D, and distance DE, describe the circle DEH, and let DA [e] be produced to the point G in the circumference thereof. Then is AG = CB.

For DG [f] = DE, and DA [g] = DC. Wherefore AG [h] = CE [i] = BC [k] = AG. Which was to be done.

The putting of the point A within or without the line BC varies the cases, but the [e] construction and demonstration are every where alike.

[a] 3 *post.*
[b] 1 *post.*
[c] 1.1.
[d] 2 *post.*
[e] 2 *post.*
[f] 15 *def.*
[g] *constr.*
[h] 3 *ax.*
[i] 15 *def.*
[k] 1 *ax.*

Scho-

Scholium.

The line AG might be taken with a pair of compaſſes, but the ſo doing anſwers to no poſtulate, as *Proclus* well intimates.

PROP. III.

Two right lines A and BC being given, from the greater BC to take away the right line BE equal to the leſſer A.

From the point B [a] draw the right line BD = A. The circle deſcribed with the center B, and diſtance BD ſhall cut off BE [b] = BD [c] = A [d] = BE. Which was to be done.

[a] 2. 1.
[b] 15 *def.*
[c] *conſtr.*
[d] 1 *ax.*

PROP. IV.

If two triangles BAC, EDF, have two ſides of the one BA, AC, equal to two ſides of the other ED, DF, each to its correſpondent ſide (that is, BA = ED, and AC = DF) and have the angle A equal to the angle D, contained under the equal right lines, they ſhall have the baſe BC equal to the baſe EF, and the triangle BAC ſhall be equal to the triangle EDF; and the remaining angles B, C ſhall be equal to the remaining angles E, F, each to each, under which the equal ſides are ſubtended.

If the point D be applied to the point A, and the right line DE placed upon the right line AB, the point E ſhall fall upon B, becauſe DE [a] = AB: alſo the right line DF ſhall fall upon AC, becauſe the angle A [a] = D: moreover the point F ſhall fall upon the point C, becauſe AC [a] = DF. Therefore the right lines EF, BC, becauſe they have the ſame terms,

[a] *hyp.*

[b] 14 ax. shall [b] agree, and so consequently are equal. Wherefore the triangles BAC, DEF, and the angles B, E, as also the angles C, F, do agree, and are equal. Which was to be demonstrated.

PROP. V.

The angles ABC, ACB, at the base of an Isosceles triangle ABC, are equal one to the other: and if the equal sides AB, AC be produced, the angles CBD, BCE, under the base, shall be equal one to the other.

[a] 3. 1. [a] Take AE = AD, and [b] joyn CD and BE.
[b] 1 post. Because in the triangles ACD, ABE, are
[c] hyp. AB [c] = AC, and AE [d] = AD, and the angle
[d] constr. A common to them both; [e] therefore is the an-
[e] 4. 1. gle ABE = ACD, and the angle AEB [e] = ADC,
[f] 3 ax. and the base BE [e] = CD. also EC [f] = DB.
[g] 4. 1. Therefore in the triangles BEC, BDC [g] shall be the angle ECB = DBC. Which was to be demonstrated.

[h] before. Also therefore the angle EBC = DCB; but
[i] 3 ax. the angle ABE [h] = ACD: therefore the angle ABC [i] = ACB. Which was to be demonstrated.

Corollarium.

Hence every equilateral triangle is also equiangular.

PROP. VI.

If two angles ABC, ACB, of a triangle ABC, be equal the one to the other, the sides AB, AC subtended under the equal angles, shall also be equal one to the other.

If the sides be not equal, let one be bigger
[a] 3. 1. than the other; suppose BA [a] > CA: make
[b] 1 post. BD = CA, and [b] draw the line CD. In

In the triangles DBC, ACB; becauſe BD [e] = CA, and the ſide BC is common, and the angle DBC [d] = ACB, the triangles DBC, ACB ſhall be equal the one to the other, a part to the whole. [f] Which is impoſſible.

[e] *ſuppoſ.* [d] *hyp.* [e] 4.1. [f] 9 *ax.*

Corollarium.

Hence every equiangular triangle is alſo equilateral.

PROP. VII.

Upon the ſame right line AB, two right lines AC, BC being drawn to any point C, two other right lines equal to the former AD, BD, each to each, (viz. AD = AC, and BD = BC) cannot be drawn from the ſame points A, B, on the ſame ſide C, to any other point than C, as ſuppoſe to D.

1 *Caſe.* If the point D be ſet in the line AC, it is plain that AD is [a] not equal to AC. [a] 9 *ax.*

2 *Caſe.* If the point D be placed within the triangle ACB, then draw the line CD, and produce BDF, and BCE. Now you would have AD = AC, then the angle ADC [b] = ACD; as alſo becauſe BD = BC, the angle FDC = [c] EGD, therefore is the angle FDC ⊏ [d] ACD, that is, the angle FDC ⊏ ADC. [d] Which is impoſſible.

[b] 5.1. [c] *ſuppoſ.* [d] 9 *ax.*

3 *Caſe.* If D falls without the triangle ACB, let CD be joyned.

Again, the angle ACD [e] = ADC, and the angle BCD [e] = BDC. [f] Therefore the angle ACD ⊏ BDC, viz. the angle ADC ⊏ BDC. Which is impoſſible. Therefore, &c.

[e] 5.1. [f] 9 *ax.*

PROP.

PROP. VIII.

If two triangles ABC, DEF have two sides AB, AC, equal to two sides DE, DF, each to each, and the base BC equal to the base EF; then the angles contained under the equal right lines shall be equal, viz. A to D.

[a] *hyp.* Because BC [a] = EF, if the base BC be laid
[b] 8 *ax.* on the base EF, [b] they will agree: therefore
[c] *hyp.* whereas AB [c] = DE, and AC = DF, the point A will fall on D, (for it cannot fall on any other point by the precedent proposition) and so the sides of the angles A and D are coincident;
[d] 8 *ax.* [d] wherefore those angles are equal. Which was to be done.

Corollaria.

1. Hence triangles mutually equilateral are
[x] 4.1. also mutually [x] equiangular.

2. Triangles mutually equilateral [x] are equal one to the other.

PROP. IX.

To bisect or divide into two equal parts a right lined angle given BAC.

[a] 3.1. [a] Take AD = AE, and draw the line DE;
[b] 1.1. upon which [b] make an equilateral triangle DFE: draw the right line AF, it shall bisect the angle.

[c] *constr.* For AD [c] = AE, and the side AF is common,
[d] 8.1. and the base DF [c] = FE: [d] therefore the angle DAF = EAF. Which was to be done.

Co-

corollarium.

Hence it appears how an angle may be cut into any equal parts, as 4, 8, 16, &c. to wit, by biſecting each part again.

The method of cutting angles into any number of equal parts required, by the Rule and Compaſs, is as yet unknown to Geometricians.

PROP. X.

To biſect a right line given AB.

Upon the line given AB [a] erect an equilateral triangle ABC; and [b] biſect the angle C with the right line CD: that line ſhall alſo biſect the line given AB. [a] 1.1. [b] 9.1.

For AC [c] = BC, and the ſide CD is common, and the angle ACD [c] = BCD: therefore AD [d] = BD. Which was to be done. [c] *conſtr.* [d] 4.1.

The practice of this and the precedent propoſition, is manifeſt by the conſtruction of prop. 1. of this book.

PROP. XI.

From a point C in a right line given AB, to erect a right line CF at right angles.

[a] Take on either ſide of the point given CD = CE, upon the right line DE [b] erect an equilateral triangle: draw the line FC, and it will be the perpendicular required. [a] 3.1. [b] 1.1.

For the triangles DFC, EFC are mutually [c] equilateral; [d] therefore the angle DCF = ECF: [e] therefore FC is perpendicular. Which was to be done. [c] *conſtr.* [d] 8.1. [e] 10 *def.*

The

The practice of this and the following is easily performed by the help of a square.

PROP. XII.

Upon an infinite right line given AB, from a point given C that is not in it, to let fall a perpendicular right line CG.

[a] 3 *post.* From the center C [a] describe a circle cutting the right line given AB in the points E and F; [b] 10. 1. then [b] bisect FE in G, and draw the right line CG, which will be the perpendicular required.

Let the lines CE, CF be drawn; the triangles [c] *constr.* EGC, FGC are mutually [c] equilateral: [d] 8. 1. therefore the angles EGC, FGC are equal, and by [e] 10 *def.* consequence right: [e] wherefore GC is a perpendicular. Which was to be done.

PROP. XIII.

When a right line AB standing upon a right line CD, maketh angles ABC, ABD; it maketh either two right angles, or two angles equal to two right.

[a] 10 *def.* If the angles ABC, ABD be equal, [a] then they make two right angles; if unequal, then from [b] 11. 1. the point B [b] let there be erected a perpendicular BE: [c] 19 *ax.* because the angle ABC [c] = to a right [d] 3 *ax.* angle + ABE, and the angle ABD [d] = to a right angle — ABE; therefore shall be ABC [e] 2 *ax.* + ABD [e] = to two right angles, + ABE — ABE = to two right angles. Which was to be demonstrated.

Corollaria.

1. Hence if one angle ABD be right, the other ABC is also right; if one acute, the other is obtuse: and so on the contrary. 2. If

2. If more right lines than one ſtand upon the ſame right line at the ſame point, all the angles ſhall be equal to two right.

3. Two right lines cutting each other make angles equal to four right ones.

4. All the angles made about one point make four right ones, as appears by corol. 2.

PROP. XIV.

If to any right line AB, and a point therein B, two right lines, not drawn on the ſame ſide, do make the angles ABC, ABD on each ſide equal to two right ones, thoſe lines CB, BD ſhall make one ſtrait line.

If you deny it, let CB, BE make one right line, then ſhall be the angle ABC + ABE [a] = to two right angles [b] = ABC + ABD. Which is [c] abſurd.

[a] 13. 1.
[b] *hyp.*
[c] 9 *ax.*

PROP. XV.

If two right lines AB, CD cut through one another, then are the two angles which are oppoſite, viz. CEB, AED equal one to the other.

For the angle AEC + CEB [a] = to two right angles = AEC + AED; [b] therefore CEB = AED. Which was to be done.

[a] 13. 1.
[b] 3 *ax.*

Scholium 1.

If to any right line GH, and in it a point A, two right lines being drawn EA, AF, and not taken on the ſame ſide, make the vertical (or oppoſite) angles D and B equal, thoſe right lines EA, AF do meet directly, and make one ſtrait line.

For

a 13.1. For two right angles are a equal to the angle D + A b = B + A: c therefore EA, AF are
b 2. in a strait line. Which was to be demonstrated.
c 14.1.

Scholium 2.

If four right lines EA, EB, EC, ED, proceeding from one point E, make the angles vertically opposite equal the one to the other, each two lines, AE, EB, and CE, ED, are placed in one strait line.

For because the angle AEC + AED + CEB
a 4 *cor.* + DEB a = to four right angles; therefore
13.1. the angle AEC + AED b = CEB + DEB =
b *hyp.* & to two right angles: c therefore CED and
2 *ax.* AEB are strait lines. Which was to be demon-
c 14.1. strated.

PROP. XVI.

One side BC of any triangle ABC being produc'd, the outward angle ACD will be greater than either of the inward and opposite angles CAB, CBA.

a 10.1. & Let the right lines AH, BE a bisect the sides
1 *post.* AC, BC; b produce EF = BE, and HI b = AH,
b 3.1. and joyn FC and IC, and produce ACG.
c *constr.* Because CE c = EA, and EF c = EB, and
d 15.1. the angle FEC d = BEA, the angle ECF e shall
e 4.1. be equal to EAB. By the like argument is the angle ICH = ABH. Therefore the whole angle
f 15.1. ACD (f BCG) g is greater than either the an-
g 9 *ax.* gle CAB or ABC. Which was to be demonstrated.

PROP.

PROP. XVII.

Two angles of any triangle ABC, which way ſoever they be taken, are leſs than two right angles.

Let the ſide BC be produced; becauſe the angle ACD + ACB [a] = to two right angles, and the angle ACD [b] ⊐ A; [c] therefore A + ACB ⊐ then two right angles. After the ſame manner is the angle B + ACB ⊐ than two right. Laſtly, the ſide AB being produced, the angle A + B will be alſo leſs than two right angles. Which was to be demonſtrated.

[a] 13. 1.
[b] 16. 1.
[c] 4 *ax.*

Corollaria.

1. Hence it follows, that in every triangle, wherein one angle is either right or obtuſe, the two others are acute angles.

2. If a right line AE make unequal angles with another right line CD, one acute AED, the other obtuſe AEC, a perpendicular AD let fall from any of its points A to the other line CD, ſhall fall on that ſide the acute angle is of.

For if AC drawn on the ſide of the obtuſe angle be a perpendicular, then in the triangle AEC ſhall AEC + ACE be greater than two right angles. * Which is contrary to the precedent propoſition. * 17. 1.

3. All the angles of an equilateral triangle, and the two angles of an Iſoſceles triangle, that are upon the baſe, are acute.

PROP. XVIII.

The greateſt ſide AC of every triangle ABC, ſubtends the greateſt angle ABC.

From

[a] 3.1. From AC [a] take away AD = AB, and joyn
[b] 5.1. BD: [b] therefore is the angle ADB = ABD;
[c] 16.1. but ADB [c] ⊐ C, therefore is ABD ⊐ C: [d] there-
[d] 9 *ax.* fore the whole angle ABG ⊐ C. After the same manner shall be ABC ⊐ A. Which was to be demonstrated.

PROP. XIX.

In every triangle ABC, under the greatest angle A is subtended the greatest side BC.

For if AB be supposed equal to BC, then will
[a] 5.1. be the angle A [a] = C, which is contrary to the Hypothesis: and if AB ⊐ BC, then shall be the
[b] 18.1. angle C [b] ⊐ A, which is against the Hypothesis: wherefore rather BC ⊐ AB. And after the same manner BC ⊐ AC. Which was to be demonstrated.

PROP. XX.

Of every triangle ABC, two sides BA, AC, any way taken, are greater than the side that remains BC.

[a] 3.1. Produce the line BA, [a] and take AD = AC,
[b] 5.1. and draw the line DC, [b] then shall the angle D
[c] 9 *ax.* be equal to ACD; [c] therefore is the whole an-
[d] 19.1. gle BCD ⊐ D: [d] therefore BD ([e] BA + AC)
[e] *constr.* & 2 *ax.* ⊐ BC. Which was to be demonstrated.

PROP. XXI.

If from the utmost points of one side BC, of a triangle ABC, two right lines BD, CD be drawn to any point within the triangle, then are both those two lines shorter than the two other sides of the triangle BA, CA, but do contain a greater angle BDC.

Let

Let BD be produced to E, then is CE + ED [a] ⊏ CD: add BD common to both, [b] then shall be BD + DE + EC ⊏ CD + BD. Again, BA + AE [a] ⊏ BE; [b] therefore BA + AC ⊏ BE + EC. Wherefore 1. BA + AC ⊏ BD + DC. 2. The angle BDC [c] ⊏ DEC [c] ⊏ A. Therefore the angle BDC ⊏ A. Which was to be demonstrated.

[a] 20.1. [b] 4 *ax.* [c] 16.1.

PROP. XXII.

To make a triangle FKG of three right lines FK, FG, GK; which shall be equal to three right lines given A, B, C: of which it is necessary that any two taken together be longer than the third.

From the infinite line DE [a] take DF, FG, GH, equal to the lines given A, B, C; then if from the centers [b] F and G, by the distances of FD and GH, two circles be drawn cutting each other in K, and the right lines KF, KG be joyned, the triangle FKG shall be made, [c] whose sides FK, FG, GK are equal to the three lines DF, FG, GH, [d] that is, to the three lines given A, B, C. Which was to be done.

[a] 3.1. [b] 3 *post.* [c] 15 *def.* [d] 1 *ax.*

PROP. XXIII.

At a point A in a right line given AB, to make a right lined angle A equal to a right lined angle given D.

[a] Draw the right line CF cutting the sides of the angle given any ways, [b] make AG = CD; upon AG [c] raise a triangle equilateral to the former CDF, so that AH be equal to DF, and GH to CF; then shall you have the angle A [d] = D. Which was to be done.

[a] 1 *post.* [b] 3.1. [c] 22.1. [d] 8.1.

PROP.

PROP. XXIV.

If two triangles ABC, DEF have two sides of the one triangle AB, AC, equal to two sides of the other triangle DE, DF, each to other, and have the angle A greater than the angle EDF, contained under the equal right lines, they shall also have the base BC greater than the base EF.

[a] 23.1. [a] Let the angle DGE be made equal to A,
[b] 3.1. and the side DG [b] = DE [c] = AC, and let EG
[c] *hyp.* and FG be joyned.
[d] *hyp.* 1 *Case.* If EG fall above EF: because AB [d] =
[e] *constr.* DE, and AC [e] = DG, and the angle A [e] =
[f] 4.1. EDG; [f] therefore is BC = EG. But because
[g] 5.1. DE [e] = DG, [g] therefore is the angle DEG =
[h] 9 *ax.* DGF, [h] therefore is the angle DFG ⊏ EGF,
and by consequence the angle EFG [h] ⊏ EGF:
[k] 19.1. [k] wherefore EG (BC) ⊏ EF.

[l] 9 *ax.* 2 *Case.* If the base EF falls in the same place with EG, [l] it is evident that EG (BC) ⊏ EF.

3 *Case.* If EG fall below EF, then because
[m] 21.1. DG + GE [m] ⊏ DF + FE, if from both DG,
DF be taken away, which are equal, EG (BC)
[n] 5 *ax.* remains [n] ⊏ EE. Which was to be demonstrated.

PROP. XXV.

If two triangles ABC, DEF have two sides AB, AC, equal to two sides DE, DF, each to other, and have the base BC greater than the base EF, they shall also have the angle A contained under the equal right lines greater than the angle D.

For if the angle A be said to be equal to D,
[a] 4.1. [a] then is the base BC = EE, which is against the Hypothesis. If it be said the angle A ⊐ D,

then

then [b] will be BC ⊐ EF, which is alſo againſt the Hypotheſis: therefore BC □ EF. Which was to be demonſtrated. [b] 14. 1.

PROP. XXVI.

If two triangles BAC, EDG have two angles of the one B, C, equal to two angles of the other E, DGE, each to his correſpondent angle, and have alſo one ſide of the one equal to one ſide of the other, either that ſide which lieth betwixt the equal angles, or that which is ſubtended under one of the equal angles; the other ſides alſo of the one, ſhall be equal to the other ſides of the other, each to his correſpondent ſide; and the other angle of the one ſhall be equal to the other angle of the other.

1 *Hypotheſis.* Let BC be equal to EG, which are the ſides that lie between the equal angles; then I ſay BA = ED, and AC = DG, and the angle A = EDG: for if it be ſaid that ED □ BA, then [a] let EH be made equal to BA, and let the line GH be drawn. [a] 3. 1.

Becauſe AB [b] = HE, and BC [c] = EG, and the angle B [c] = E, therefore ſhall be the angle EGH [d] = C [e] = DGE. [f] Which is abſurd. [b] *ſuppoſ.* [c] *hyp.* [d] 4. 1. [e] *hyp.* [f] 9 *ax.*

After the ſame manner let AC be equal to DG, [d] then will the angle A be equal to EDG.

2 *Hypotheſis.* Let AB be equal to ED; then I ſay BC = EG, and AC = DG, and the angle A = EDG: for if EG be greater than BC, make EI = BC, and joyn the line DI. Now becauſe AB [g] = ED, and BC [h] = EI, and the angle B [g] = E; therefore will be the angle EID [k] = C [l] = EGD. [m] Which is abſurd. [g] *hyp.* [h] *ſuppoſ.* [k] 4. 1. [l] *hyp.* [m] 16. 1.

Therefore BC = EG, and ſo as before AC = DG, and the angle A = EDG. Which was to be demonſtrated.

PROP.

PROP. XXVII.

If a right line EF falling upon two right lines AB, CD, make the alternate angles AEF, DFE, equal the one to the other, then are the right lines AB, CD parallel.

If AB, CD be said not to be parallel, produce them till they meet in G; which being supposed, the outward angle AEF will be [a] greater than the inward angle DFE, to which it was equal by the Hypothesis. Which is repugnant.

[a] 16. 1.

PROP. XXVIII.

If a right line EF falling upon two right lines AB, CD, make the outward angle AGE, of the one line, equal to CHG, the inward and opposite angle of the other on the same side; or make the inward angles on the same side AGH, CHG equal to two right angles, then are the right lines AB, CD parallel.

1 *Hypothesis.* Because by Hypothesis the angle AGE = CHG, [a] therefore are the alternate angles BGH, CHG equal: and [b] therefore are AB and CD parallel.

[a] 15. 1.
[b] 27. 1.

2 *Hypothesis.* Because by Hypothesis the angle AGE + CHG = to two right, [a] = AGH + BGH; [b] thence is CHG = BGH: and [c] therefore AB, CD are parallel. Which was to be demonstrated.

[a] 13. 1.
[b] 3 *ax.*
[c] 27. 1.

PROP. XXIX.

If a right line EF fall upon two parallels AB, CD, it will both make the alternate angles DHG, AGH equal each to other, and the outward angle

BGE

BGE equal to the inward and oppoſite angle on the ſame ſide DHE; as alſo the inward angles on the ſame ſide AGH, CHG equal to two right angles.

It is evident that AGH + CHG = to two right angles; [a] otherwiſe AB, CD would not be parallel, which is contrary to the Hypotheſis. But moreover the angle DHG + CHG [b] = to two right; therefore is DHG [c] = AGH [d] = BGE. Which was to be demonſtrated.

[a] 13 ax.
[b] 13. 1.
[c] 3 ax.
[d] 15. 1.

Corollarium.

Hence it follows, that every parallelogram AC, having one angle right A, the reſt are alſo right.

For A + B [a] = to two right angles: therefore whereas A is right, [b] muſt B be alſo right. By the ſame argument are C and D right angles.

[a] 29. 1.
[b] 3 ax.

PROP. XXX.

Right lines AB, CD parallel to one and the ſame right line EF, are alſo parallel the one to the other.

Let GI cut the three right lines given any ways; then becauſe AB, EF are parallel, will be the angle AGI [a] = EHI: alſo becauſe CD and EF are parallel, will be the angle EHI [a] = DIG. [b] Therefore the angle AGI = DIG: [c] whence AB and CD are parallel. Which was to be demonſtrated.

[a] 29. 1.
[b] 1 ax.
[c] 27 1.

PROP. XXXI.

From a point given A to draw a right line AE, parallel to a right line given BC.

From the point A draw a right line AD to any point of the given right line; with which

at

[a] 23. 1. [b] 27. 1. at the point thereof [a] A make an angle DAE = ADC: [b] then will AE and BC be parallel. Which was to be done.

PROP. XXXII.

Of any triangle ABC, one side BC being drawn out, the outward angle ACD shall be equal to the two inward opposite angles A, B: and the three inward angles of the triangle A, B, ACB, shall be equal to two right angles.

[a] 31. 1. [b] 29. 1. [c] 2 *ax.* [d] 19 *ax.* From C [a] draw CE parallel to BA; then is the angle A [b] = ACE, and the angle B [b] = ECD: therefore A + B [c] = ACE + ECD [d] = ACD. Which was to be demonstrated.

[e] 13. 1. [f] 1 *ax.* I affirm ACD + ACB [e] = to two right angles: [f] therefore A + B + ACB = to two right angles. Which was to be demonstrated.

Corollaria.

1. The three angles of any triangle taken together, are equal to three angles of any other triangle taken together. From whence it follows,

2. That if in one triangle two angles (taken severally or together) be equal to two angles of another triangle (taken severally or together) then is the remaining angle of the one, equal to the remaining angle of the other. In like manner if two triangles have one angle of the one, equal to one of the other; then is the summ of the remaining angles of the one triangle, equal to the summ of the remaining angles of the other.

3. If one angle in a triangle be right, the other two are equal to a right one. Likewise that

that angle in a triangle, which is equal to the other two, is it self a right angle.

4. When in an Isosceles the angle made by the equal sides is right, the other two upon the base are each of them half a right angle.

5. An angle of an equilateral triangle makes two third parts of a right angle; for ½ of two right angles is equal to ⅔ of one.

Scholium.

By the help of this proposition you may know how many right angles the inward and outward angles of a right lined figure make, as may appear by these two following theorems.

Theorema 1.

All the angles of a right lined figure do together make twice as many right angles, abating four, as there are sides of the figure.

From any point within the figure let right lines be drawn to all the angles of the figure, which shall resolve the figure into as many triangles as there are sides of the figure: wherefore whereas every triangle affords two right angles, all the triangles taken together will make up twice as many right angles as there are sides. But the angles about the said point within the figure make up four right; therefore if from the angles of all the triangles you

take away the angles which are about the said point, the remaining angles, which make up the angles of the figure, will make twice as many right angles, abating four, as there are sides of the figure. Which was to be demonstrated.

Corollarium.

Hence all right lined figures of the same species have the summs of their angles equal.

Theorema 2.

All the outward angles of any right lined figure, taken together, make up four right angles.

For all the several inward angles of a figure, with the several outward angles of the same apart, make two right angles; therefore all the inward angles together, with all the outward, make twice as many right angles as there are sides of the figure. But (as it was now shewn) all the inward angles with four right, make twice as many right as there are sides of the figure; therefore the outward angles are equal to four right angles. Which was to be demonstrated.

Corollarium.

All right lined figures of whatsoever species have the summs of their outward angles equal.

PROP. XXXIII.

If two equal and parallel lines AB, CD be joyned together with two other right lines AC, BD, then are those lines also equal and parallel.

Draw

Draw a line from C to B: now becauſe AB and CD are parallel, and the angle ABC [a] = BCD: and alſo by Hypotheſis AB = CD, and the ſide CB common; therefore is AC [b] = BD, and the angle ACB [b] = DBC: [c] whence alſo AC, BD are parallel. [a] 29.1. [b] 4.1. [c] 27.1.

PROP. XXXIV.

In parallelograms, as ABCD, the oppoſite ſides AB, CD, and AC, BD, are equal each to other; and the oppoſite angles A, D and ABD, ACD are alſo equal: and the diameter BC biſects the ſame.

Becauſe AB, CD [a] are parallel, [b] therefore is the angle ABC = BCD. Alſo becauſe AC, ED are [a] parallel. [b] therefore is the angle ACb = CBD; [c] therefore the whole angle ACD = AED. After the ſame manner is A = D. Moreover, becauſe the angles ABC, ACB lie at each end of the ſide CB, and are equal to BCD, CBD, [d] therefore is AC = BD, and AB [d] = CD; and ſo the triangle ABC = CBD. Which was to be demonſtrated. [a] *hyp.* [b] 29.1. [c] 2 *ax.* [d] 26.1.

Scholium.

Every four ſided figure ABDC, having the oppoſite ſides equal, is a parallelogram.

For by 8.1 the angle AEC = BCD; [a] wherefore AB, CD are parallel. In like manner is the angle BCA = CBD; [a] wherefore AC, BD are alſo parallel: [b] therefore ABCD is a parallelogram. Which was to be demonſtrated. [a] 27.1. [b] 35 *def.* 1.

From hence may be learned how to draw more readily a parallel CD, to a right line given AB, through a point aſſigned C.

Take in the line AB any point, as E; with the centers E and C at any distance draw two equal circles EF, CD; with the center F, and the space of EC draw a circle FD, which shall cut the former circle CD in the point D: then shall the line drawn CD be parallel to AB; for as it was before demonstrated, CEFD is a parallelogram.

PROP. XXXV.

Parallelograms BCDA, BCFE, which stand upon the same base BC, and between the same parallels AF, BC, are equal one to the other.

[a] 34. 1. For AD [a] = BC [a] = EF, add DE common to
[b] 2 *ax.* both; [b] then is AE = DF: but also AB [a] =
[c] 29. 1. DC, and the angle A [c] = CDF: [d] therefore is
[d] 4. 1. the triangle ABE = DCF, take away DGE com-
[e] 3 *ax.* mon to both triangles, [e] then is the trapezium
ABGD = EGCF: add BGC common to both,
[f] 2 *ax.* [f] then is the parallelogram ABCD = EBCF.
Which was to be demonstrated.

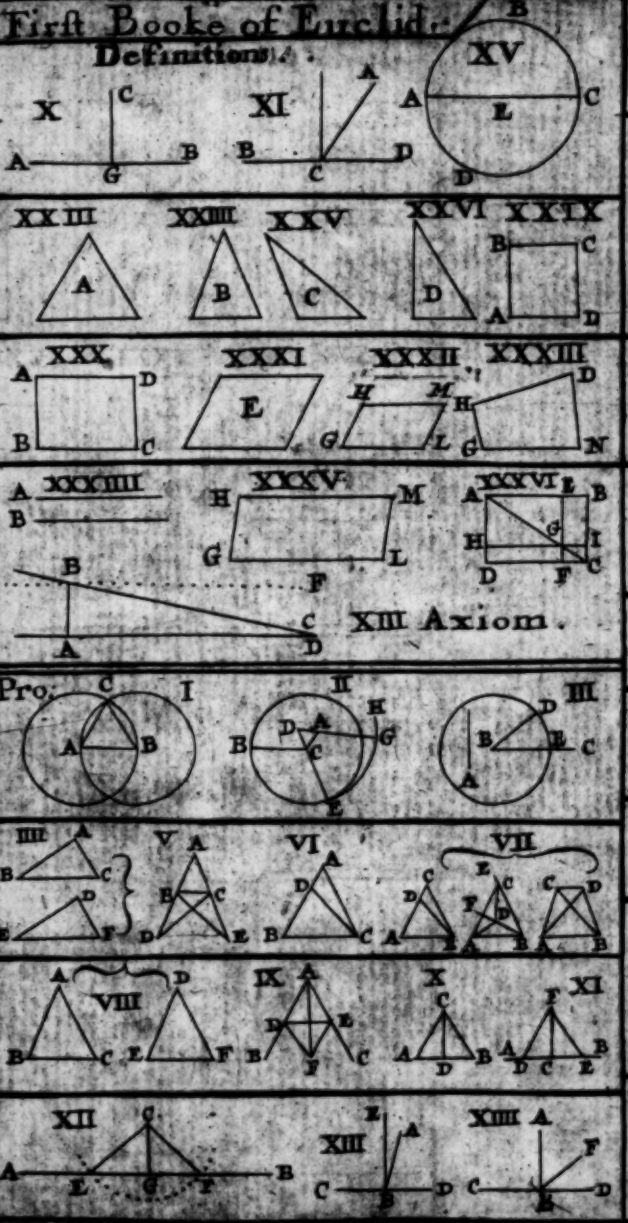
The demonstration of any other cases is not unlike, but much more plain and easie.

Scholium.

If the side AB of a right angled parallelogram AECD, be conceived to be carried along perpendicularly through the whole line EC, or EC through the whole line AB, the area or content of the rectangle ABCD shall be produc'd by that motion. Hence a rectangle is said to be made by the drawing or multiplication of two contiguous sides: for example sake, let AB be supposed 4 foot, and EC 3; draw 3 into 4, there will be produced 12 square feet for the area of the rectangle. This

First Booke of Euclid.

Definitions.



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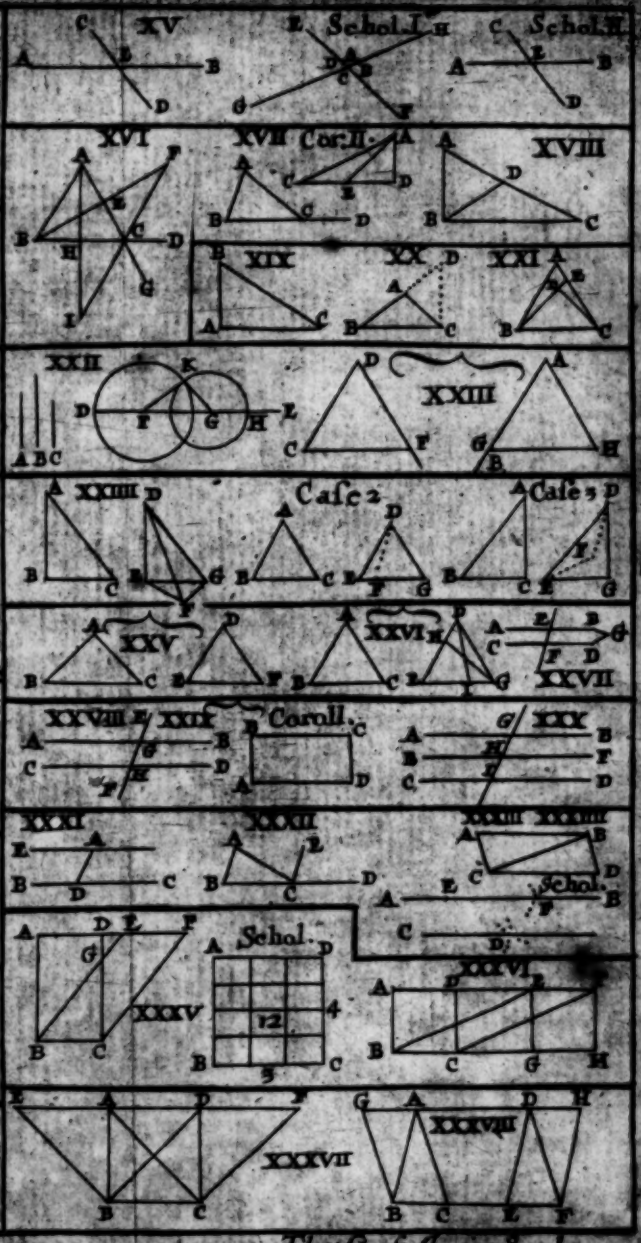

XV
Schol. I
Schol. II
XVI
XVII Cor. II.
XVIII
XIX
XX
XXI
XXII
XXIII
XXIV
Case 2
Case 3
XXV
XXVI
XXVII
XXVIII
XXIX
Coroll.
XXX
XXXI
XXXII
XXXIII XXXIV
Schol.
XXXV
XXXVI
XXXVII
Tho: Cross Junior Sculp.

This being ſuppoſed, the dimenſion of any parallelogram (* EBCF) is found out by this theorem: for the area thereof is produced from the altitude BA drawn into the baſe BC: ſo the area of the parallelogram AC = EBCF, is made by the drawing of BA into BC: therefore, &c.

* *See the figure of* Prop. 35.

PROP. XXXVI.

Parallelograms BCDA, GHFE, ſtanding upon equal baſes BC, GH, and betwixt the ſame parallels AF, BH, are equal one to the other.

Draw BE and CF: becauſe BC [a] = GH [b] = EF; [c] therefore is BCFE a parallelogram. Whence the parallelogram BCDA [d] = BCFE [d] = GHFE. Which was to be demonſtrated.

[a] *hyp.* [b] 34. 1. [c] 33. 1. [d] 35. 1.

PROP. XXXVII.

Triangles BCA, BCD, ſtanding upon the ſame baſe BC, and between the ſame parallels BC, EF, are equal one to the other.

[a] Draw BE parallel to CA, [a] and CF parallel to BD; then is the triangle BCA [b] = ½ of the parallelogram BCAE [c] = ½ BDFC [b] = BCD. Which was to be demonſtrated.

[a] 31. 1. [b] 34. 1. [c] 35. 1. & 7 ax.

PROP. XXXVIII.

Triangles BCA, EFD, ſet upon equal baſes BC, EF, and between the ſame parallels GH, BF, are equal the one to the other.

Draw BG parallel to CA, and FH parallel to ED: then is the triangle BCA [a] = ½ the parallelogram BCAG [b] = ½ EDHF [c] = EFD. Which was to be demonſtrated.

[a] 34. 1. [b] 36. 1. & 7 ax. [c] 34. 1.

Scholium.

If the base BC be greater than EF, 'tis evident that the triangle BAC ⊏ EDF: and so on the contrary.

PROP. XXXIX.

Equal triangles BCA, ECD, standing on the same base BC, and on the same side, they are also between the same parallels AD, BC.

If you deny it, let another line AF be parallel to EC, and let CF be drawn; then is the triangle CBF [a] = CBA [b] = CBD. [c] Which is absurd.

[a] 37. 1.
[b] *hyp.*
[c] 9 *ax.*

PROP. XL.

Equal triangles BCA, EFD, standing upon equal bases BC, EF, and on the same side, they are betwixt the same parallels.

If you deny it, let another line AH be parallel to EF, and let FH be drawn: then is the triangle EFH [a] = BCA [b] = EFD. [c] Which is absurd.

[a] 38. 1.
[b] *hyp.*
[c] 9 *ax.*

PROP. XLI.

If a parallelogram ABCD have the same base EC with the triangle BCE, and be between the same parallels AE, BC, then is the parallelogram ABCD double to the triangle BCE.

Let the line AC be drawn; then is the triangle BCA [a] = BCE: therefore in the parallelogram ABCD [b] = 2BCA [c] = 2BCE. Which was to be demonstrated.

[a] 37. 1.
[b] 34. 1.
[c] 6 *ax.*

Scho-

Scholium.

From hence may the area of any triangle BCE be found: for whereas the area of the parallelogram AECD is produced by the altitude drawn into the baſe, therefore ſhall the area of a triangle be produced by half the altitude drawn into the baſe, or half the baſe drawn into the altitude: as if the baſe BC be 8, and the altitude 7, then is the area of the triangle BCE 28.

PROP. XLII.

To make a parallelogram ECGF equal to a triangle given ABC, in an angle equal to a right lined angle given D.

Through A[a] draw AG parallel to BC; [a] 31. 1.
[b] make the angle ECG = D; [c] biſect the baſe [b] 23. 1.
BC in E, and draw EF parallel to CG, then is [c] 10. 1.
the problem ſolved.

For AE being drawn, the angle ECG is equal [d] 38. 1.
to D by conſtruction, and the triangle BAC[d] [e] 41. 1.
= 2AEC[e] = parallelogram ECGF. Which was to be done.

PROP. XLIII.

In every parallelogram ABCD, the complements DG, GB of thoſe parallelograms HE, FI, which ſtand about the diameter, are equal one to the other. See Fig. def. 36.

For the triangle ACD[a] = ACB, and the [a] 34. 1.
triangle AGH[a] = AGE, and the triangle
GCF[a] = GCI: [b] therefore the parallelogram [b] 3 *ax.*
DG = BG. Which was to be demonſtrated.

PROP. XLIV.

Upon a right line given A, to make a parallelogram FL in a right lined angle given C, equal to a triangle given B.

[a] 42.1. [a] Make a parallelogram FD = to the triangle B, so that the angle GFE may be equal to C. Produce GF till FH be equal to the line given A. Through H [b] draw IL parallel to EF, which [b] 31.1. let DE produced meet in the point I. Let DG drawn forth meet with a right line drawn from I in the point K Through K [b] draw KL parallel to GH, with which let EF drawn out meet at M, and IH at L. Then shall FL be the parallelogram required.

[c] 43.1. [d] 15.1. For the parallelogram FL [c] = FD = B, [d] and the angle MFH = GFE = C. Which was to be done.

PROP. XLV.

Upon a right line given FG, to make a parallelogram FL equal to a right lined Figure given ABCD, at a right lined angle given F.

[a] 44.1. Resolve the right lined figure given into triangles BAD, BCD, then [a] make a parallelogram FH = BAD, so that the angle F may be equal to E. FI being produced, [a] make on HI the parallelogram IL = BCD. Then is the parallelogram FL [b] = FH + IL [c] = ABCD. Which was to be done.

[b] 19 ax. [c] *constr.*

Scholium.

Hence is easily found the excess HK, whereby any right lined figure A, exceeds a less right lined

lined figure B; namely if to ſome right line GF both be applied, parallelogram FL = A, and FH = B.

PROP. XLVI.

Upon a right line given AD to deſcribe a ſquare AC.

[a] Erect two perpendiculars AB, DC, [b] equal to the line given AD; then joyn BC, and the thing required is done. [a] 11.1. [b] 3.1.

For, whereas the angle A + D [c] = two right, [d] therefore are AB, DC parallel. But they are alſo [e] equal, [f] therefore AD, BC are both parallel and equal; therefore the figure AC is a parallelogram and equilateral. Moreover, the angles are all right, [g] becauſe one, A, is right, therefore AC is a ſquare. Which was to be done. [c] *conſtr.* [d] 28.1. [e] *conſtr.* [f] 33.1. [g] *ſch.* 29. 1 [h] 29 *def.*

After the ſame manner you may eaſily deſcribe a rectangle contained under two right lines given.

PROP. XLVII.

In right angled triangles BAC, the ſquare BE, which is made of the ſide BC, that ſubtends the right angle BAC, is equal to both the ſquares BG, CH, which are made of the ſides AB, AC, containing the right angle.

Joyn AE and AD, and draw AM parallel to CE.

Becauſe the angle DBC [a] = FBA, add the angle ABC common to them both; then is the angle ABD = FBC. Moreover, AB [b] = FB, and BD [b] = BC; [c] therefore is the triangle ABD = FBC. But the parallelogram BM [d] = 2ABD [a] 12 *ax.* [b] 29 *def.* [c] 4.1. [d] 41.1.

d 4.1. 2ABD, and the parallelogram d BG = 2FBC (for GAC is one right line by Hypothesis, and
e 6 ax. 14.1) e therefore is the parallelogram BM =
f 2 ax. BG. By the same way of argument is the parallelogram CM = CH; therefore is the f whole BE = BG + CH. Which was to be demonstrated.

Scholium.

This most excellent and useful theorem hath deserved the title of *Pythagoras* his theorem, because he was the inventor of it. By the help of which the addition and subtraction of squares are performed; to which purpose serve the two following problems.

Problem 1.

Andr. Tacq. *To make one square equal to any number of squares given.*

Let three squares be given, whereof the sides
a 11.1. are AB, BC, CE. a Make the right angle FBZ, having the sides infinite, and on them transfer
b 47.1. BA and EC; joyn AC, then is ACq b = ABq + BCq. Then transfer AC from B to X, and CE the third side given from B to E; joyn EX.
c 2 ax. b Then is EXq = EBq (CEq) + BXq (ACq) c = CEq + ABq + ECq. Which was to be done.

Problem 2.

Two unequal right lines being given AB, BC, to make a square equal to the difference of the two squares of the given lines AB, BC.

From the center B, at the distance of BA describe a circle; and from the point C erect a perpendicular CE meeting with the circumference

rence in E, and draw BE. [a] Then is BEq (BAq) = BCq + CEq. [b] Therefore BAq — BCq = CEq. Which was to be done.

[a] 47.1.
[b] 3 *ax.*

Problem 3.

Any two sides of a right angled triangle ABC being known, to find out the third.

Let the sides AB, AC, encompassing the right angle, be the one 6 foot, the other 8. Therefore whereas ACq + AEq = 64 + 36 = 100 = BCq, thence is BC = $\sqrt{100}$ = 10. 47.1.

Otherwise let the sides AB, EC be known, the one 6 foot, the other 10. Therefore since BCq — AEq = 100 — 36 = 64 = ACq, thence is AC = $\sqrt{64}$ = 8. Which was to be done. 47.1.

PROP. XLVIII.

If the square made upon one side BC of a triangle, be equal to the squares made on the other sides of the triangle AB, AC, then the angle BAC comprehended under those two other sides of the triangle AB, AC, is a right angle. See Fig. prop. 20.

Draw to the point A in AC a perpendicular line DA = AB, and joyn CD.

Now is [a] CDq = ADq + ACq = AEq + ACq = BCq. * Therefore is CD = BC. And therefore the triangles CAB, CAD are mutually equilateral. Wherefore the angle CAB [b] = CAD [c] = a right angle. Which was to be demonstrated.

[a] 47.1.
* *See the following theorem.*
[b] 8.1.
[c] *constr.*

Scholium.

We assumed in the demonstration of the last proposition CD = BC, because CDq was equal to

to BCq: our assumption we prove by the following theorem.

Theorema.

The squares AF, CG of equal right lines AB, CD, are equal one to the other: and the sides IK, LM of equal squares NK, PM, are equal one to the other.

1 *Hypothesis.* Draw the diameters EB, HD, then it is evident that AF is [a] equal to the triangle EAB twice taken, and [b] equal to the triangle HCD twice taken, and equal to [a] CG. Which was to be demonstrated.

[a] 34. 1.
[b] 4. 1. & 6 *ax.*

2 *Hypothesis.* If it may be let LM be greater than IK; make LT = IK, and let LS be [a] equal to LTq: therefore is LS [b] = NK [c] = LQ. [d] Which is absurd.

[a] 46. 1.
[b] 1 *part.*
[c] *hyp.*
[d] 9 *ax.*

Corollarium.

After the same manner any rectangles equilateral one to another, are demonstrated to be also equal.

The End of the First Book.

THE SECOND BOOK OF Euclid's *Elements*.

Definitions.

I. EVery right angled parallelogram ABCD is ſaid to be contained under two right lines AB, AD, comprehending a right angle.

Therefore when you meet with ſuch as theſe, the rectangle under BA, AD, or for ſhortneſs ſake the rectangle BAD or BA × AD (or Z A for Z × A) the rectangle meant, is that which is contained under the right lines BA, AD ſet at right angles.

II. In every parallelogram FHIK, any one of thoſe parallelograms which are about the diameter, together with its two complements, is called a Gnomon.

As the parallelogram FB + BI + GA (EHM) is a gnomon: and likewiſe the parallelogram FB + BI + EM (GKA) is a gnomon.

PROP.

PROP. I.

If two right lines AF, AB be given, and one of them AB divided into as many parts or ſegments as you pleaſe; the rectangle comprehended under the two whole right lines AB, AF, ſhall be equal to all the rectangles contained under the whole line AF, and the ſeveral ſegments AD, DE, EB.

[a] 11. I. [a] Set AF perpendicular to AB. Through F [a] draw an infinite line FG perpendicular to AF. From the points D, E, B erect perpendiculars DH, EI, BG. Then is AG a rectangle comprehended under AF, AB, and is [b] equal to the rectangles AH, DI, EG; that is, (becauſe DH, EI, [b] 19 ax. I. AF [c] are equal) to the rectangles under AF, AD, [c] 34. I. under AF, DE, under AF, EB. Which was to be demonſtrated.

Scholium.

If two right lines given be both divided into how many parts ſoever, the product of the whole multiplied into it ſelf, ſhall be the ſame with that of the parts multiplied into themſelves.

For let Z be = A + B + C, and Y = D + E: [a] 1. 2. then becauſe DZ [a] = DA + DB + DC, and EZ [a] = EA + EB + EC, and YZ [a] = DZ + [b] 2 ax. EZ, [b] ſhall ZY be = DA + DB + DC + EA + EB + EC. Which was to be demonſtrated.

From hence is underſtood the manner of multiplying compounded right lines into compounded: for you muſt take all the rectangles of the parts, and they will preſent you with the rectangle of the wholes.

But whenſoever in the multiplication of lines into themſelves, you meet with theſe ſigns — in-

intermingled with these +, you must also have particular regard to the signs: for of + multiplied into — ariseth —; but of — into — ariseth +.

E.g. Let +A be to be multiplied into B—C; then because +A is not affirmed of all B, but onely of a part of it, whereby it exceeds C, therefore AC must remain denied; so that the product will be AB—AC. Or thus; because B consists of the parts C and B—C, * thence AB=AC+A×B—C; take away AC from either, then AB—AC=A×B—C. In like manner if —A be to be multiplied into B—C, then seeing by reason of the sign —, that A is not denied of all B, but onely of so much as it exceeds C, therefore AC must remain affirmed: whence the product will be —AB+AC. Or thus; because AB * = AC+A×B—C; take away all throughout, and there will be —AB=—AC—A—×B—C; add AC to either, and there will be —AB+AC=—A× B—C. * I.2. * I.2.

This being sufficiently understood, the nine following propositions, and innumerable others of that kind, arising from the comparing of lines multiplied into themselves (which you may find done to your hand in *Vieta* and other Analytical Writers) are demonstrated with great facility, by reducing the matter for the most part to almost a simple work.

Furthermore, * it appears that the product of any magnitude multiplied into the parts of any number, is equal to the product of the same multiplied into the whole number: As 5A+7A=12A, and 4\×5A+4A×7A=4A×12A. Wherefore what is here delivered of the multiplying of right lines into themselves, the same * 19 *ax.*

same may be understood of the multiplying of numbers into themselves. So that whatsoever is affirmed concerning lines in the nine following theorems, holds good also in numbers; seeing they all immediately depend on, and are derived from this first.

PROP. II.

If a right line Z be divided any wise into two parts, the rectangles comprehended under the whole line Z, and each of the segments A, E, are equal to the square made of the whole line Z.

I say that ZA + ZE = Zq. For take B = Z;
[a] 1.2. [a] then is BA + BE = BZ: that is, (because B = Z) ZA + ZE = Zq. Which was to be demonstrated.

PROP. III.

If a right line Z be divided any wise into two parts, the rectangle comprehended under the whole line Z, and one of the segments E, is equal to the rectangle made of the segments A, E, and the square described on the said segment E.

I say, ZE = AE + Eq; for EZ = E × A + E =
[a] 1.2. [a] EA + Eq.

PROP. IV.

If a right line Z be cut any wise into two parts, the square made of the whole line Z, is equal both to the squares made of the segments A, E, and to twice a rectangle made of the parts A, E.

[a] 3.2. I say that Zq = Aq + Eq + 2AE; for ZA [a] =
[b] 2.2. Aq + AE, and ZE [a] = Eq + EA. Therefore
[c] 1 ax. whereas ZA + ZE [b] = Zq, [c] thence is Zq = Aq + Eq

-+Eq+2AE. Which was to be demonſtrated.

Otherwiſe thus: Upon the right line AB make the ſquare AD, and draw the diameter EB: through C, the point wherein the line AB is divided, draw the perpendicular CF; and through the point G draw HI parallel to AB.

Becauſe the angle EHG = A is a right angle, and AEB is [d] half a right, [e] therefore is the remaining angle HGE half a right angle. Therefore is HE [f] = HG [g] = EF [g] = AC; ſo that HF [h] is the ſquare of the right line AC. After the ſame manner is CI proved to be CBq. Therefore AG, GD are rectangles under AC, CB. Wherefore the whole ſquare AD [k] = ACq -+ CBq +2ACB. Which was to be demonſtrated.

[d] 4 *cor.* 32. 1.
[e] 32. 1.
[f] 6. 1.
[g] 34. 1.
[h] 29 *def.* 1
[k] 19 *ax.* 1.

Corollaria.

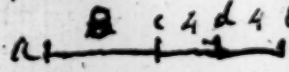
1. Hence it appears that the parallelograms, which are about the diameter of a ſquare, are alſo ſquares themſelves.

2. That the diameter of any ſquare biſects its angles.

3. That if A = ¼Z, then is Zq = 4Aq, and Aq = ¼Zq. As on the contrary, if it be ſo that Zq = 4Aq, then is A = ¼Z.

PROP. V.

If a right line AB be cut into equal parts AC, CB, and into unequal parts AD, DB; the rectangle comprehended under the unequal parts AD, DB, together with the ſquare that is made of the difference of the parts CD, is equal to the ſquare that is made of the half line CB.



12 4 8 I
4 4 8
48 16 64
16
48

I ſay that CBq = ADB + CDq.

For theſe are all equal:

CBq.
[a] 4.2. [a] CDq + CDB + DBq + CDB.
[b] 3.2. CDq + [b] CBD ([c] AC × BD) + CDB.
[c] *hyp.* CDq + [d] ADB.
[d] 1.2.

This theorem is ſomewhat differently expreſs'd, and more eaſily demonſtrated thus: A rectangle made of the ſum and the difference of two right lines A, E, is equal to the difference out of them.

For if A + E be multiplied into A — E, *ſch. 1.2. *there ariſeth Aq — AE + EA — Eq = Aq — Eq. Which was to be demonſtrated.

Scholium.

If the line AB be divided otherwiſe, *viz.* nearer to the point of biſection in E, then is AEB ⊏ ADB.

[a] 5.2. & 3 *ax.* For AEB[a] = CBq — CEq, and ADB[a] = CBq — CDq. Therefore whereas CDq ⊏ CEq, thence is AEB ⊏ ADB. Which was to be demonſtrated.

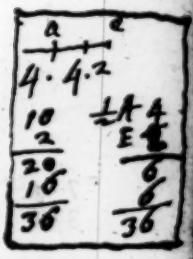
Corollaria.

1. Hence is ADq + DBq ⊏ AEq + EBq: for [b] 4.2. ADq + DBq + 2ADB[b] = ABq[b] = AEq + EBq + 2AEB. Therefore becauſe 2AEB ⊏ 2ADB, thence is ADq + DBq ⊏ AEq + EBq. Which was to be demonſtrated.

[c] 3 *ax.* 2. Hence is ADq + DBq — AEq[c] — EBq = 2AEB — 2ADB.

PROP.

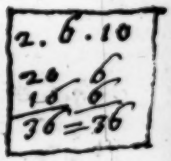
PROP. VI.



If a right line A be divided into two equal parts, and another right line E added to the same directly in one right line, then the rectangle comprehended under the whole, and the line added (viz. A+E) and the line added E, together with the square which is made of ½ the line A, is equal to the square of ½A+E taken as one line.

I say that ¼Aq (a Q.½A) +AE+Eq = Q. ½A+E: a for Q.½A+E = ¼Aq +Eq+AE. a 4 & 3 cor. 4. 2.
Which was to be demonstrated.

corollarium.



Hence it follows, that if three right lines E, E+½A, E+A be in arithmetical proportion, then the rectangle contained under the extreme terms E, E+A, together with the square of the difference ½A, is equal to the square of the middle term E+½A.

PROP. VII.

If a right line Z be divided anywise into two parts, the square of the whole line Z, together with the square made of one of the segments E, is equal to a double rectangle comprehended under the whole line Z, and the said segment E, together with the square made of the other segment A. See Fig. prop. 2.

I say that Zq+Eq = 2ZE+Aq; for Zq = a Aq+Eq+2AE, and 2ZE b = 2Eq+2AE. a 4. 2. b 3. 2.
Therefore Zq+Eq = c Aq+2Eq+2AE = c 2 ax. Aq+2ZE. Which was to be demonstrated.

Co-

Corollarium.

Hence it follows, that the ſquare of the difference of any two lines Z, E, is equal to the ſquares of both the lines, leſs by a double rectangle comprehended under the ſaid lines.

[c] For Zq + Eq — 2ZE = Aq = Q. Z — E.

[c] 7.2. & 3 ax.

PROP. VIII.

See Fig. prop. 2.

If a right line Z be divided any wiſe into two parts, the rectangle comprehended under the whole line Z, and one of the ſegments E four times, together with the ſquare of the other ſegment A, is equal to the ſquare of the whole line Z and the ſegment E, taken as one line Z + E.

I ſay that 4ZE + Aq = Q. Z + E. For 2ZE [a] = Zq + Eq — Aq. Therefore 4ZE + Aq = Zq + Eq + 2ZE [b] = Q. Z + E. Which was to be demonſtrated.

[a] 7.2. & 3 ax.
[b] 4.2.

PROP. IX.

See Fig. prop. 5.

If a right line AB be divided into equal parts AC, CB, and into unequal parts AD, DB; then are the ſquares of the unequal parts AD, DB together, double to the ſquare of the half line AC, and to the ſquare of the difference CD.

I ſay, that ADq + DBq = 2ACq + 2CDq: for ADq + DBq [a] = ACq + CDq + 2ACD + DBq. But 2ACD ([b] 2BCD) + DBq [c] = CBq (ACq) + CDq. [d] Therefore ADq + DBq = 2ACq + 2CDq. Which was to be demonſtrated.

[a] 4.2.
[b] *hyp.*
[c] 7.2.
[d] 2 *ax.*

This may be otherwiſe delivered, and more eaſily demonſtrated thus: The aggregate of the

the ſquares made of the ſum, and the difference of two right lines A, E, is equal to the double of the ſquares made from thoſe lines.

For Q: A + E [a] = Aq + Eq + 2AE, and Q: A − E [b] = Aq + Eq − 2AE. Theſe added together make 2Aq + 2Eq. Which was to be demonſtrated.

[a] 4.2
[b] *Cor.* 7.2.

PROP. X.

If a right line A be divided into two equal parts, and another line be added in a right line with the ſame, then is the ſquare of the whole line, together with the added line (as being one line) together with the ſquare of the added line E, double to the ſquare of ½A and the added line E, taken as one line.

See Fig. prop. 6.

I ſay that Eq + Q. A + E, i. e. [a] Aq + 2Eq + 2AE = 2Q. ½A + 2Q. ½A + E: for 2Q. ½A [b] = ½Aq. And 2Q. ½A + E [c] = ½Aq + 2Eq + 2AE. Which was to be demonſtrated.

[a] 4.2.
[b] *cor.* 4.2.
[c] 4.2.

PROP. XI.

To cut a right line given AB in a point G, ſo that the rectangle comprehended under the whole line AB, and one of the ſegments BG, ſhall be equal to the ſquare that is made of the other ſegment AG.

Upon AB [a] deſcribe the ſquare AC; [b] biſect the ſide AD in E, and draw the line EB. From the line EA produced take EF = EB. On AF make the ſquare AH. Then is AH = AB × BG.

[a] 46. 1.
[b] 10. 1.

For HG being drawn out to I, the rectangle DH + EAq [c] = EBq [d] = EFq [e] = BAq + EAq. Therefore is DH [f] = BAq equal to the ſquare AC. Take away AI common to both, then remains the ſquare AH = GC, that is, AGq = AB × BG. Which was to be done.

[c] 6.2.
[d] *conſtr.*
[e] 47. 1.
[f] 3 *ax.*

Scho-

Scholium.

* See 6.13. This propoſition cannot be performed by numbers, * for there is no number that can be ſo divided, that the product of the whole into one part, ſhall be equal to the ſquare of the other part.

PROP. XII.

In obtuſe angled triangles ABC, the ſquare that is made of the ſide AC ſubtending the obtuſe angle ABC, is greater than the ſquares of the ſides BC, AB, that contain the obtuſe angle ABC, by a double rectangle contained under one of the ſides BC, which are about the obtuſe angle ABC, on which ſide produced the perpendicular AD falls, and under the line BD, taken without the triangle from the point on which the perpendicular AD falls, to the obtuſe angle ABC.

I ſay that ACq = CBq + ABq + 2CB × BD.

For theſe are all equal:
- ACq.
- [a] CDq + ADq.
- [b] CBq + 2CBD + BDq + ADq.
- CBq + 2CBD + [c] ABq.

[a] 47.1. [b] 4.2. [c] 47.1.

Scholium.

Hence the ſides of any obtuſe angled triangle ABC being known, the ſegment BD intercepted betwixt the perpendicular AD and the obtuſe angle ABC; as alſo the perpendicular it ſelf AD ſhall be eaſily found out.

Thus: let AC be 10, AB 7, CB 5; then is ACq 100, ABq 49, CBq 25; and ABq + CBq = 74. Take that out of 100, then will 26 remain

main for 2CBD. Wherefore CBD ſhall be 13; divide this by CB5, there will $2\frac{1}{5}$ be found for BD. Whence AD will be found out by the 47.1. (x)

PROP. XIII.

In acute angled triangles ABC, the ſquare made of the ſide AB, ſubtending the acute angle ACB, is leſs than the ſquares made of the ſides AC, CB, comprehending the acute angle ACB by a double rectangle contained under one of the ſides BC, which are about the acute angle ACB, on which the perpendicular AD falls, and under the line DC taken within the triangle from the perpendicular AD to the acute angle ACB.

I ſay that ACq + BCq = ABq + 2BCD.

For theſe are equal:
- ACq + BCq.
- [a] ADq + DCq + BCq. — [a] 47.1.
- [b] ADq + BDq + 2BCD. — [b] 7.2.
- [c] ABq + 2BCD. — [c] 47.1.

Corollarium.

Hence the ſides of an acute angled triangle ABC being known, you may find out the ſegment DC intercepted betwixt the perpendicular AD, and the acute angle ABC, as alſo the perpendicular it ſelf AD.

Let AB be 13, AC 15, BC 14. Take ABq (169) from ACq + BCq; that is, from 225 + 196 = 421. Then remains 252 for 2BCD; wherefore BCD will be 126. Divide this by BC 14, then will 9 be found out for DC. From whence it follows, AD = $\sqrt{225 - 81}$ = 12. (y)

PROP.

PROP. XIV.

To find a ſquare ML equal to a right lined figure given A.

[a] Make the rectangle DB = A, and produce the greater ſide thereof DC to F, ſo that CF = CB. [b] Biſect DF in G, about which as the center at the diſtance of GF deſcribe the circle FHD, and draw out CB till it touch the circumference in H; then ſhall be CHq = [*] ML = A.

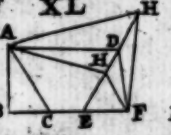
For let GH be drawn; then is A [c] = DB [c] = DCF [d] = GFq — GCq [e] = HGq [e] = ML. Which was to be done.

[a] 45. I.
[b] 10. I.
[*] 46. I.
[c] *conſtr.*
[d] 5. 2. & 3 *ax.*
[e] 47. I. & 3 *ax.*

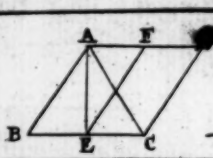
The End of the Second Book.

Prop: XXXIX

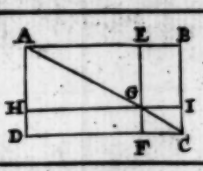
XL



XLI. *Euclide Lib: 1*

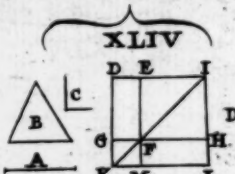


XLII



XLIII

XLIV



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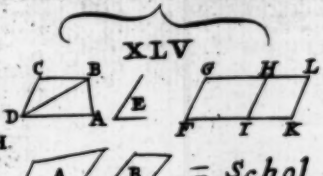

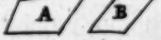

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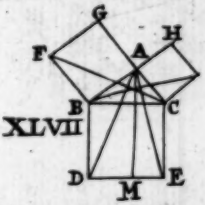
XLVI

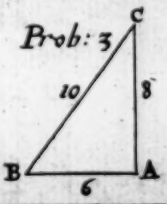


XLVII

Prob: 1

Prob: 2

Prob: 3



XLVIII

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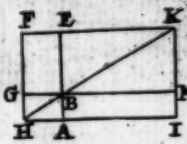
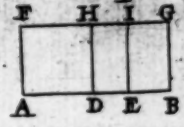
The Second Book of Euclid.

Def: I

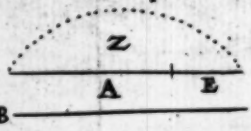
Def: II



Prop: I



II. III. & IV.



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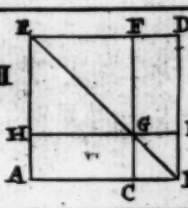

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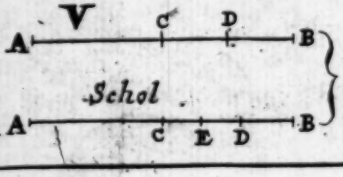

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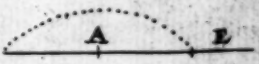
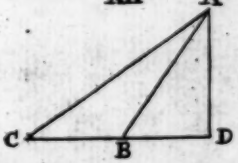


XI

XII



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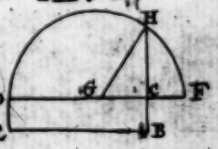

Place this at the end of the Second Book.

Tho: Cross Junior Sculp.

THE THIRD BOOK OF Euclid's *Elements.*

Definitions.

I. EQual circles GABC, HDEF are ſuch whoſe diameters are equal, or from whoſe centers right lines drawn GA, HD are equal.

II. A right line AB is ſaid to touch a circle FED, when touching the ſame, and being produced, it cutteth it not. The right line FG cuts the circle FED.

III. Circles DAC, ABE (and alſo FBG, ABE) are ſaid to touch one to the other, which touch, but cut not one the other. The circle BFG cuts the circle FGH.

IV. In a circle GABD right lines FE, KL are ſaid to be equally diſtant from the center, when perpendiculars GH, GN drawn from the center G to them are equal. And that line BC is ſaid to be fartheſt diſtant from it, on whom the greater perpendicular GI falls.

V. A ſegment of a circle ABC is a figure contained under a right line AC, and a portion of the circumference of a circle ABC.

VI. An angle of a ſegment CAB, is that angle which is contained under a right line CA, and an arch of a circle AB.

VII. An angle ABC is ſaid to be in a ſegment ABC, when in the circumference thereof ſome point B is taken, and from it right lines AB, CB, drawn to the ends of the right line AC, which is the baſe of the ſegment: then the angle ABC contained under the adjoyned lines AB, CB is ſaid to be an angle in a ſegment.

VIII. But when the right lines AB, BC, comprehending the angle ABC, do receive any periphery of the circle ADC, then the angle ABC is ſaid to ſtand upon that periphery.

IX. A ſector of a circle ADB is when an angle ADB is ſet at the center D of that circle; namely that figure ADB, comprehended under the right lines AD, BD containing the angle, and the part of the circumference received by them AB.

X. Like ſegments of a circle ABC, DEF, are thoſe which conclude equal angles ABC, DEF, or in whom the angles ABC, DEF are equal.

PROP. I.

To find the center F of a circle given ABC.

Draw a right line AC any wiſe in the circle, which biſect in E, through E draw a perpendicular DB, and biſect the ſame in F: the point F ſhall be the center.

If you deny it, let G a point without the line DB be the center (for it cannot be in the line BD,

BD, ſince that cannot be divided equally in any point but F.) Let the lines GA GC, GE be drawn. Now if G be the center, [a] then is GA =GC, and AE=EC by conſtruction, and the ſide GE common. [b] Therefore are the angles GEA, GEC equal, and [c] conſequently right. [d] Therefore the angle GEC=FEC. [e] Which is abſurd.

[a] 15 *def.* 1.
[b] 8. 1.
[c] 10 *def.* 1.
[d] 12 *ax.*
[e] 9 *ax.*

Corollarium.

Hence if a right line BD biſect any right line AC in a circle at right angle, the center ſhall be in the line BD that cuts the other.

The center of a circle is eaſily found out, by applying the top of a ſquare to the circumference thereof. And. Thaet.

For if the right line DE, that joyns the points D, E, in which the ſides of the ſquare QD, QE cut the circumference, be biſected in A, the point A ſhall be the center. The demonſtration whereof depends upon Prop. 31. of this Book.

PROP. II.

If in the circumference of a circle CAB, any two points A, B be taken, the right line AB, which joyns thoſe two points, ſhall fall within the circle.

Take in the right line AB any point D. From the center C draw CA, CD, CB. Becauſe CA [a] =CB, therefore is the angle A [b] = B. But the angle CDB [c] ⊏ A, therefore is CDB ⊏ B, therefore CB [d] ⊏ CD. But CB onely reaches the circumference, therefore CD comes not ſo far: wherefore the point D is within the circle. The ſame may be proved of any other

[a] 15 *def.* 1.
[b] 5. 1.
[c] 16. 1.
[d] 19. 1.

point in the line AB. And therefore the whole line AB falls within the circle. Which was to be demonstrated.

Corollarium.

Hence if a right line toueh a circle, so that it cut it not, it touches but in one point.

PROP. III.

If in a circle EABC, a right line BD drawn through the center, bisect any other line AG not drawn through the center, it shall also cut it at right angles: and if it cut it at right angles, it shall also bisect the same.

From the center E let the lines EA, EC be drawn.

1 *Hypothesis.* Because AF[a] = FC, and EA[b] = EC, and the side EF common; the angles EFA, EFC[c] shall be equal, and[d] consequently right. Which was to be demonstrated.

2 *Hypothesis.* Because EFA[e] = EFC, and the angle EAF[f] = ECF, and the side EF common; [g] therefore is AF = FC. Therefore AC is cut into two equal parts. Which was to be demonstrated,

a *hyp.*
b 15 *def.* 1
c 8. 1.
d 10 *def.* 1
e *hyp.* & 12 *ax.*
f 5. 1.
g 26. 1.

Corollarium.

Hence in any equilateral or Isosceles triangle, if a line drawn from the vertical angle bisect the base, that line is perpendicular to it. And on the contrary, a perpendicular drawn from the vertical angle bisects the base.

PROP.

PROP. IV.

If in a circle ACD two right lines AB, CD cut through one another, yet neither of them paſs through the center E, then both thoſe lines do not mutually cut each other in half.

For if one line paſs through the center, it appears that it cannot be biſected by the other; becauſe by Hypotheſis, the other does not paſs through the center.

If neither of them paſs through the center, then from the center E draw EF: now if AB, CD were both biſected, in F, then [a] would the angles EFB, EFD be both right, and conſequently equal. [b] Which is abſurd.

[a] 3.3.

[b] 9 *ax.*

PROP. V.

If two circles BAC, BDC cut one the other, they ſhall not have the ſame center E.

For otherwiſe the lines EB, EDA drawn from E the common center, would be DE [a] = EB [a] = EA. [b] Which is abſurd.

[a] 15 *def.* 1

[b] 9 *ax.*

PROP. VI.

If two circles BAC, BDC inwardly touch one the other in B, they have not one and the ſame center F.

For otherwiſe the right lines FB, FDA drawn from the center F, would be FD [a] = FB [a] = FA. [b] Which is abſurd.

[a] 15 *def.* 1.

[b] 9 *ax.*

PROP. VII.

If in A B t' é diameter of a circle ſome point G be aken, which is not the center of the circle, and from

that point certain right lines GC, GD, GE fall on the circle, the greateſt line ſhall be that GA in which is the center F, the leaſt, the remainder of the ſame line GB. And of all the other lines the line GC, neareſt to that which was drawn through the center, is always greater than any line farther removed GD; and onely two lines equal to one another GE, GH, fall upon the circle from the ſame point, on each ſide of the leaſt GB, or of the greateſt GA.

From the center F draw the right lines FC, FD, FE, and [a] make the angle BFH = BFE.

[a] 23. I.

[a] 20. I. 1. GF-|-FC (that is, GA) [a] ⊏ GC. Which was to be demonſtrated.

[b] 15 *def.* I. [c] 9 *ax.* [d] 24. I. 2. The ſide FG is common, and FC [b] = FD, and the angle GFC [c] ⊏ GFD: [d] wherefore the baſe GC ⊏ GD.

[e] 20. I. [f] 5 *ax.* 3. FB (FE) [e] ⊐ GE-|-GF. Therefore FG, which is common, being taken away from both, there remains BG ⊐ EG.

[g] *conſtr.* [h] 4. I. 4. The ſide FG is common, and FE = FH, and the angle BFH [g] = BFE. [h] Therefore is GE = GH. But that no other line GD from the point G can be equal to GE or GH, is already proved. Which was to be demonſtrated.

PROP. VIII.

If ſome point A be taken without a circle, and from that point be drawn certain right lines AI, AH, AG, AF to the circle; and of thoſe one AI be drawn through the center K, and the others any wiſe. Of all thoſe lines that fall on the concave of the circumference, that is the greateſt AI which is drawn through the center: and of the others, that which is neareſt (AH) to the line that paſſes through the center, is greater than that which is more diſtant AG. But of all thoſe lines that fall on the convex part of the circle, the

the leaſt is that AB which is drawn from the point A to the diameter IB: and of the others, that (AC) which is neareſt to the leaſt, is leſs than that which is farther diſtant: AD. And from that point there can be onely two equal right lines AC, AL drawn, which ſhall fall on the circumference on each ſide of the leaſt line AB, or of the greateſt AI.

From the center K draw the right lines KH, KG, KF, KC, KD, KE, and make the angle AKL =AKC.

1. AI (AK-|-KH) [a] ⊏ AH. [a] 20. 1.

2. The ſide AK is common, and KH=KG, and the angle AKH⊏ AKG. [b] Therefore the baſe AH⊏ AG. [b] 24. 1.

3. KA [c] ⊐ KC -|- CA. From hence take away KC, KB that are equal, then will remain AB [d] ⊐ AC. [c] 20. 1. [d] 5 *ax.*

4. AC-|-CK [e] ⊐ AD-|-DK. From thence take away CK, DK that are equal, then remains AC [f] ⊐ AD. [e] 21. 1. [f] 5 *ax.*

5. The ſide KA is common, and KL=KC, and the angle AKL [g] = AKC. [h] Therefore KA =CA. But that no other line could be drawn equal to theſe, was proved before. Therefore, &c. [g] *conſtr.* [h] 4. 1.

PROP. IX.

If in a circle BCK a point A be taken, and from that point more than two equal right lines AB, AC, AK drawn to the circumference, then is that point A the center of the circle.

For [a] from no point without the center can more than two right lines equal be drawn to the circumference. Therefore A is the center. Which was to be demonſtrated. [a] 7. 3.

PROP. X.

A circle IAKBL cannot cut another circle IEKFL in more than two points.

Let one circle, if it may be, cut the other in three points I, K, L, and IK, KL being joyned, let them be biſected in M and N. [a] Both circles have their centers in their perpendiculars MC, NH, and in the interſection of thoſe perpendiculars, which is O. Therefore the circles that cut each other have the ſame center. Which is falſe by Prop. 5. 3.

[a] *cor.* 1. 3.

PROP. XI.

If two circles GADE, FABC touch one the other inwardly, and their centers be taken G, F: a right line FG joyning their centers, and produced, ſhall cut the circumference in A, the point of contact of the circles.

If it can be, let the right line FG produced cut the circles in ſome other point than A; ſo that not FGA, but FGDB ſhall be a right line. Let the line GA be drawn. Now becauſe GD [a] = GA, and GB [b] ⌝ GA (ſince the right line FGB paſſes through F the center of the greater circle) therefore is GB ⌝ GD. [c] Which is abſurd.

[a] 15 *def.* 1.
[b] 7. 3.
[c] 9 *ax.*

PROP. XII.

If two circles ACD, BCE touch one the other outwardly, the right line AB, which joyns their centers A, B, ſhall paſs through the point of contact C.

If it may be, let ADEB be a right line cutting the circles, not in the point of contact C, but

but in the points D,E: draw AC, CB. Then is AD + EB (AC + CB) [a] ⊏ ADEB. [b] Which is absurd.

[a] 20. 1.
[b] 9 *ax.*

PROP. XIII.

A circle CAF cannot touch a circle BAH in more points than one A, whether it be inwardly or outwardly.

1. Let one circle (if it can be) touch another in two points A,H; [a] then will the right line CB that joyns the centers, if it be produced, fall as well in A as H. Now because CH [b] = CA, and BH ⊏ CH, therefore is BA ([c] BH) ⊏ CA. [d] Which is absurd.

[a] 11. 3.
[b] 15 *def.* 1.
[c] 15 *def.* 1.
[d] 9 *ax.*

2. If it be said to touch outwardly in the points E and F, then draw the line EF, [e] which will be in both circles: therefore those circles cut one the other. Which is against the Hypothesis.

[e] 2. 3.

PROP. XIV.

In a circle EABC equal right lines AC, BD are equally distant from the center E: and right lines AC, BD, which are equally distant from the center, are equal among themselves.

From the center E draw the perpendiculars EF, EG, [a] which will bisect the lines AC, BD; joyn EA, EB.

[a] 3 3.

1 *Hypothesis.* AC = BD: therefore AF [b] = BG. But also EA = EB, therefore FEq [c] = EAq — AFq = EBq — BGq [c] = EGq. [d] Therefore FE = EG.

[b] 7 *ax.*
[c] 47. 1. & 3 *ax.*
[d] *Sch.* 48. 1

2 *Hypothesis.* EF = EG. Therefore AFq [e] = EAq — EFq = EBq — EGq = BGq. Therefore AF [d] = GB, and [e] consequently AC = BD. Which was to be demonstrated.

[e] 6 *ax.*

PROP. XV.

In a circle GABC the greateſt line is AD the diameter; and of all other lines, that line FE, which is neareſt to the center G, is greater than any line BC farther diſtant from it.

1. Draw GB and GC, the diameter AD
[a] 15 *def.* 1. ([a] GB + GC) [b] ⊏ BC.
[b] 20. 1. 2. Let the diſtance GI be ⊏ GH, take GN = GH. Through the point N draw KL perpendicularly to GI; joyn GK, GL. Becauſe GK = GB, and GL = GC, and the angle KGL ⊏ BGC, [c] therefore is KL (FE) ⊏ BC. Which was
[c] 24. 1. to be demonſtrated.

PROP. XVI.

A line CD drawn from the extreme point of the diameter HA of a circle BALH, perpendicular to the ſaid diameter, ſhall fall without the circle: and between the ſame right line and the circumference cannot be drawn another line AL. And the angle of the ſemicircle BAI, is greater than any right lined acute angle BAL: and the remaining angle without the circumference DAI is leſs than any right lined angle.

1. From the center B to any point F, in the right line AC, draw the right line BF. The
[a] 19. 1. ſide BF ſubtending the right angle BAF is [a] greater than the ſide BA, which is oppoſite to the acute angle BFA. Therefore whereas BA (EG) reaches to the circumference, BF ſhall reach farther; and ſo the point F. And for the ſame reaſon is any other point of the line AC placed without the circle.

2. Draw

2. Draw BE perpendicular to AL. The ſide BA oppoſite to the right angle BEA is [b] greater than the ſide BE, which ſubtends the acute angle BAE: therefore the point E, and ſo the whole line EA falls within the circle. [b] 19. 1.

3. Hence it follows, that any acute angle, to wit EAD, is greater than the angle of contact DAI; and that any acute angle BAL is leſs than the angle of a ſemicircle BAI. Which was to be demonſtrated.

Corollaria.

Hence a right line drawn from the extremity of the diameter of a circle, and at right angles, is a tangent to the ſaid circle.

From this propoſition are gathered many paradox and wonderful conſectaries, which you may meet with in the Interpreters.

PROP. XVII.

From a point given A to draw a right line AG, which ſhall touch a circle given DBC.

From D the center of the circle given let a line DA, cutting the circumference in B, be drawn to the point given A; from the center D deſcribe another circle through the point A: and from B draw a perpendicular to AD, which ſhall meet with the circle AE in the point E, and draw ED meeting with the circle BC in the point C. Then the line drawn from A to C ſhall touch the circle DBC.

For DB [a] = DC, and DE [a] = DA, and the angle D is common: [b] therefore the angle ACD = EBD and right, [c] therefore AC touches the circle in C. Which was to be done. [a] 15 *def.* 1. [b] 4. 1. [c] *cor.* 16. 3.

PROP.

PROP. XVIII.

If any right line AB touch a circle FEDC, and from the center to the point of contact E a right line FE be drawn; that line FE ſhall be perpendicular to the tangent AB.

If you deny it, let ſome other line FG be drawn from the center F perpendicular to the tangent, and [a] cutting the circle in D. Therefore whereas the angle FGE is ſaid to be right, [b] thence is the angle FEG acute: [c] ſo that FE (FD) ⊏ FG. [d] Which is abſurd.

[a] 2 *def.* 3. & 16. 3.
[b] *cor.* 17. 1.
[c] 19. 1.
[d] 9 *ax.*

PROP. XIX.

If any right line AB touch a circle, and from the point of contact C a right line CE be erected at right angles to the tangent, the center of the circle ſhall be in the line CE ſo erected.

If you deny it, let the center be without the line CE in the point F, and from F to the point of contact let FC be drawn. Therefore [*] the angle FCB is right, and [a] conſequently equal to the angle ECB, which was right by Hypotheſis. [b] Which is abſurd.

[*] 18. 3.
[a] 11 *ax.*
[b] 9 *ax.*

PROP. XX.

In a circle DABC the angle BDC at the center is double of the angle BAC at the circumference, when the ſame arch of the circle BC is the baſe of the angles.

Draw the diameter ADE, the outward angle BDE [a] = DAB + DBA [b] = 2DAB. Likewiſe the angle EDC = 2DAC. Therefore in the firſt caſe [c] the whole angle BDC = 2BAC, and in

[a] 32. 1.
[b] 5. 1.
[c] 2 *ax.*

in the third caſe the [d] remaining angle BDC = 2BAC. Which was to be demonſtrated. [d] 20 *ax.* & 5. 1.

PROP. XXI.

In a circle EDAC, the angles DAC and DBC, which are in the ſame ſegment, are equal one to the other.

1 *Caſe.* If the ſegment DABC be greater than a ſemicircle, from the center E draw ED, EC, then is twice the angle A [a] = E [a] = 2B. Which was to be demonſtrated. [a] 20. 3.

2 *Caſe.* If the ſegment be leſs than a ſemicircle, then is the ſum of the angles of the triangle ADF equal to the ſum of the angles of the triangle BCF. From each let AFD be taken away [b] equal to BFC, and ADB [c] = ACB be likewiſe taken away, then remains DAC = DBC. Which was to be demonſtrated. [b] 15. 1. [c] *By the* 1 *caſe.*

PROP. XXII.

The angles ADC, ABC of a quadrilateral figure ABCD deſcribed in a circle, which are oppoſite one to the other, are equal to two right angles.

Draw AC, BD, the angles ABC + BCA + BAC [a] = two right. But BDA [b] = BCA, and BDC [b] = BAC, [c] therefore ABC + ADC = two right angles. Which was to be demonſtrated. [a] 32. 1. [b] 21. 13. [c] 1 *ax.*

Corollaria.

1. Hence if one ſide * AB of a quadrilateral deſcribed in a circle be produced, the external angle EBC is equal to the internal angle ADC, which is oppoſite to that ABC, which is adjacent to EBC. As appears by 13. 1. and 3 *ax.* * *See the Fig. of the following Scholium.*

2. A

2. A circle cannot be deſcribed about a rhombus becauſe its oppoſite angles are greater or leſs than two right angles.

Scholium.

If in a quadrilateral ABCD the angles A and C, which are oppoſite, be equal to two right, then a circle may be deſcribed about that quadrilateral.

For a circle will paſs through any three angles (as ſhall appear by 5.4.) I ſay that ſhall paſs through A, the fourth alſo of ſuch a quadrilateral. For if you deny it, let the circle paſs through F. Therefore the right lines BF, FD, BD being drawn, the angle C + F [a] = two right [b] = C + A: wherefore A [c] is equal to F. [d] Which is abſurd.

[a] 22.3. [b] *hyp.* [c] 3 *ax.* [d] 21.1.

PROP. XXIII.

Two like and unequal ſegments of circles ABC, ADC cannot be ſet on the ſame right line AC, and the ſame ſide thereof.

For if they are ſaid to be like, draw the line CB cutting the circumferences in D and B, joyn AB and AD: becauſe the ſegments are ſuppoſed like, [a] therefore is the angle ADC = ABC. [b] Which is abſurd.

[a] 10 *def.* 3. [b] 16.1.

PROP. XXIV.

Like ſegments of circles ABC, DEF, upon equal right lines AC, DF, are equal one to the other.

The baſe AC being laid on the baſe DF will agree with it, becauſe AC = DF. Therefore the ſegment ABC ſhall agree with the ſegment DEF, (for otherwiſe it ſhall fall either within

or

or without, and if ſo, [a] then the ſegments are not like, which is contrary to the Hypotheſis; or at leaſt it ſhall fall partly within and partly without, and ſo cut in three points, [b] which is abſurd.) [c] Therefore the ſegment ABC = DEF. Which was to be demonſtrated.

[a] 23.3.
[b] 10.3.
[c] 8 *ax.*

PROP. XXV.

A ſegment of a circle ABC being given, to deſcribe the whole circle whereof that is a ſegment.

Let two right lines be drawn AB, BC, which biſect in the points D and E. From D and E draw the perpendiculars DF, EF, meeting in the point F. I ſay this point ſhall be the center of the circle.

For the center ſhall be as well in [a] DF as EF, therefore it muſt be in the common point F. Which was to be done.

[a] *cor.* 1.3.

PROP. XXVI.

In equal circles GABC, HDEF, equal angles ſtand upon equal parts of the circumference AC, DF; whether thoſe angles be made at the centers G, H, or at the circumferences B, E.

Becauſe the circles are equal, therefore is GA = HD, and GC = HF: alſo by Hypotheſis the angle G = H: [a] therefore AC = DF. Moreover the angle B [b] = ½ G [c] = ½ H [b] = E. [d] Therefore the ſegments ABC, DEF are like, and [e] conſequently equal; [f] whence the remaining ſegments alſo AC, DF are equal. Which was to be demonſtrated.

[a] 4.1.
[b] 20.3.
[c] *hyp.*
[d] 10 *def.* 3.
[e] 24.3.
[f] 3 *ax.*

Scho-

Scholium.

In a circle ABCD let an arch AB be equal to DC, then ſhall AD be parallel to BC. For the right line AC being drawn, the angle ACB
[a] 26.3. [a] = CAD. Wherefore by 27.1. the ſaid ſides are parallel.

PROP. XXVII.

In equal circles GABC, HDEF, the angles ſtanding upon equal parts of the circumference AC, DF, are equal between themſelves, whether they be made at the centers G, H, or at the circumferences B, E.

For, if it be poſſible, let one of the angles AGC be ☐ DHF, and make AGI = DHF:
[a] 26.3. thence is the arch AI [a] = DF [b] = AC. [c] Which
[b] *hyp.* is abſurd.
[c] 9 *ax.*

Scholium.

A right line EF, which being drawn from A the middle point of any periphery BC, toucheth the circle, is parallel to the right line BC ſubtending the ſaid periphery.

From the center D draw a right line DA to the point of contact A, and joyn DB, DC.

The ſide DG is common, and DB = DC, and
[a] 27. 3. the angle BDA [a] = CDA, (becauſe the arches
[b] *hyp.* BA, CA are [b] equal) therefore the angles at
[c] 4.1. the baſe DGB, DGC are [c] equal, and [d] conſe-
[d] 10 *def.* 1. quently right: but the inward angles GAE,
[e] *hyp.* GAF are alſo [e] right; [f] therefore BC, EF are
[f] 28.1. parallel. Which was to be demonſtrated.

PROP.

PROP. XXVIII.

In equal circles GABC, HDEF, equal right lines AC, DF cut off equal parts of the circumference, the greateſt ABC equal to the greateſt DEF, and the leaſt AIC to the leaſt DKF.

From the centers G,H draw GA,GC, and HD,HF.

Becauſe GA=HD, and GC=HF, and AC=DF,[b] therefore is the angle G=H:[c] whence the arch AIC=DKF,[d] and ſo the remaining arch ABC=DEF. Which was to be demonſtrated.

[a] *hyp.*
[b] 8.1.
[c] 26.3.
[d] 3 *ax.*

But if the ſubtended line AC be ⌐ or ¬ than DF, then in like manner will the arch AC be ⌐ or ¬ than DF.

PROP. XXIX.

In equal circles GABC, HDEF, equal right lines AC, DF ſubtend equal peripheries ABC, DEF.

Draw the lines GA,GC, and HD,HF: becauſe GA=HD, and GC=HF, and (becauſe the arches AC,DF are[a] equal) the angle G=H.[c] Therefore is the baſe AC=DF. Which was to be demonſtrated.

[a] *hyp.*
[b] 27.3.
[c] 4.1.

This and the three precedent propoſitions may be underſtood alſo of the ſame circle.

PROP. XXX.

To cut a periphery given ABC into two equal parts.

Draw the right line AC, and biſect it in D; from D draw a perpendicular DB, meeting with the arch in B, it ſhall biſect the ſame.

For

For joyn AB and CB, the ſide DB is common,
[a] *conſtr.* and AD [a] = DC, and the angle ADB [b] =
[b] 12 *ax.* CDB. [c] Therefore AB = BC, [d] whence the arch
[c] 4.1. AB = BC. Which was to be done.
[d] 28.3.

PROP. XXXI.

In a circle the angle ABC, which is in the ſemicircle, is a right angle; but the angle, which is in the greater ſegment BAC, is leſs than a right angle; and the angle which is in the leſſer ſegment BFC is greater than a right angle. Moreover, the angle of the greater ſegment is greater than a right angle, and the angle of the leſſer ſegment is leſs than a right angle.

From the center D draw DB, becauſe DB =
[a] 5.1. DA; therefore is the angle A [a] = DBA, and
[b] 2 *ax.* the angle DCB [a] = DBC, [b] therefore the an-
[c] 32.1. gle ABC = A + ACB [c] = EBC: [d] ſo that
[d] 10 *def.* 1. ABC and EBC are right angles. Which was to be demonſtrated.

[e] *Cor.* 17.1. [e] therefore BAC is an acute angle. [f] Which
[f] 22.3. was to be demonſtrated.

And further, whereas BAC + BFC = two right, therefore BFC is an obtuſe angle.

Laſtly, the angle contained under the right line CB, and the arch BAC is greater than the right angle ABC; but the angle made by the right line CB, and the periphery of the leſſer
[g] 9 *ax.* ſegment BFC [g] is leſs than the right angle ABC. Which was to be demonſtrated.

Scholium.

In a right angled triangle ABC, if the hypothenuſe (or ſubtended line) AC be biſected in D, a circle drawn from the center D through the point A,

A, ſhall alſo paſs through the point B. As you may eaſily demonſtrate from this propoſition and 21. I.

PROP. XXXII.

If a right line AB touch a circle, and from the point of contact be drawn a right line CE cutting the circle, the angles ECB, ECA, which it makes with the tangent line, are equal to thoſe angles EDC, EFC, which are made in the alternate ſegments of the circle.

Let CD, the ſide of the angle EDC, be perpendicular to AB (a for it's to the ſame purpoſe) b therefore CD is the diameter, c therefore the angle CED in a ſemicircle is a right angle, d and therefore the angle D+DCE= to a right angle e =ECB+DCE. f Therefore the angle D=ECB. Which was to be demonſtrated.

Now whereas the angle ECB+ECA g = two right h =D+F, from both of theſe take away ECB and D, which are equal, k then remains ECA=F. Which was to be demonſtrated.

a 26. 3.
b 19. 3.
c 31. 3.
d 32. 1.
e *conſtr.*
f 3 *ax.*
g 13. 1.
h 22. 3.
k 3 *ax.*

PROP. XXXIII.

Upon a right line AB to deſcribe a ſegment of a circle AIEB, which ſhall contain an angle AIB equal to a right lined angle given C.

a Make the angle BAD=C, through the point A draw the line AE perpendicular to HD. At one end of the line given AB make an angle ABF=BAF, one ſide whereof let it cut the line AE in F. From the center F through the point A deſcribe a circle, which ſhall paſs through

a 23. 1.

For joyn AB and CB, the ſide DB is common,
[a] *conſtr.* and AD [a] = DC, and the angle ADB [b] =
[b] 12 *ax.* CDB. [c] Therefore AB = BC, [d] whence the arch
[c] 4.1. AB = BC. Which was to be done.
[d] 28.3.

PROP. XXXI.

In a circle the angle ABC, which is in the ſemicircle, is a right angle; but the angle, which is in the greater ſegment BAC, is leſs than a right angle; and the angle which is in the leſſer ſegment BFC is greater than a right angle. Moreover, the angle of the greater ſegment is greater than a right angle, and the angle of the leſſer ſegment is leſs than a right angle.

From the center D draw DB, becauſe DB =
[a] 5.1. DA; therefore is the angle A [a] = DBA, and
[b] 2 *ax.* the angle DCB [a] = DBC, [b] therefore the an-
[c] 32.1. gle ABC = A + ACB [c] = EBC: [d] ſo that
[d] 10 *def.* 1. ABC and EBC are right angles. Which was to be demonſtrated.
[e] *Cor.* 17.1. [e] therefore BAC is an acute angle. [f] Which
[f] 22.3. was to be demonſtrated.

And further, whereas BAC + BFC = two right, therefore BFC is an obtuſe angle.

Laſtly, the angle contained under the right line CB, and the arch BAC is greater than the right angle ABC; but the angle made by the right line CB, and the periphery of the leſſer
[g] 9 *ax.* ſegment BFC [g] is leſs than the right angle ABC. Which was to be demonſtrated.

Scholium.

In a right angled triangle ABC, if the hypothenuſe (or ſubtended line) AC be biſected in D, a circle drawn from the center D through the point A,

A, ſhall alſo paſs through the point B. As you may eaſily demonſtrate from this propoſition and 21. I.

PROP. XXXII.

If a right line AB touch a circle, and from the point of contact be drawn a right line CE cutting the circle, the angles ECB, ECA, which it makes with the tangent line, are equal to thoſe angles EDC, EFC, which are made in the alternate ſegments of the circle.

Let CD, the ſide of the angle EDC, be perpendicular to AB ([a] for it's to the ſame purpoſe) [b] therefore CD is the diameter, [c] therefore the angle CED in a ſemicircle is a right angle, [d] and therefore the angle D+DCE= to a right angle [e] =ECB+DCE. [f] Therefore the angle D=ECB. Which was to be demonſtrated.

[a] 26. 3. [b] 19. 3. [c] 31. 3. [d] 32. 1. [e] *conſtr.* [f] 3 *ax.*

Now whereas the angle ECE+ECA [g] = two right [h] =D+F, from both of theſe take away ECB and D, which are equal, [k] then remains ECA=F. Which was to be demonſtrated.

[g] 13. 1. [h] 22. 3. [k] 3 *ax.*

PROP. XXXIII.

Upon a right line AB to deſcribe a ſegment of a circle AIB, which ſhall contain an angle AIB equal to a right lined angle given C.

[a] Make the angle BAD=C, through the point A draw the line AE perpendicular to HD. At one end of the line given AB make an angle ABF=BAF, one ſide whereof let it cut the line AE in F. From the center F through the point A deſcribe a circle, which ſhall paſs through

[a] 23. 1.

through B, (becauſe the angles FBA [b] = FAB, and [c] therefore FB = FA) and AIB is the ſegment ſought. For becauſe HD is perpendicular to the diameter AE, it therefore [d] touches the circle HD, which AB cuts. And therefore the angle AIB [e] = BAD [f] = C. Which was to be done.

[b] *conſtr.*
[c] 6.1.
[d] *cor.* 16.3.
[e] 32.3. *v*
[f] *conſtr.*

PROP. XXXIV.

From a circle given ABC to cut off a ſegment ABC, containing an angle B, equal to a right lined angle given.

[a] Draw a right line EF, which ſhall touch the circle given in A, [b] let AC be drawn alſo making an angle FAC = D. This line ſhall cut off ABC, containing an angle B [c] = CAF [d] = D. Which was to be done.

[a] 17.3.
[b] 23.1.
[c] 32.3.
[d] *conſtr.*

PROP. XXXV.

If in a circle FBCA two right lines AB, DC cut each other, the rectangle comprehended under the ſegments AE, EB of the one, ſhall be equal to the rectangle comprehended under the ſegments CE, ED of the other.

1 *Caſe.* If the right lines cut one the other in the center, the thing is evident.

2 *Caſe.* If one line AB paſs through the center F, and biſect the other line CD, then draw FD. Now the rectangle AEB + FEq [a] = FBq [b] = FDq [c] = EDq + FEq [d] = CED + FEq. [e] Therefore the rectangle AEB = CED. Which was to be demonſtrated.

[a] 5.2.
[b] *ſch.* 48.1.
[c] 47.1.
[d] *hyp.*
[e] 3 *ax.*

3 *Caſe.* If one of the lines AB be the diameter, and cut the other line CD unequally, biſect CD by FG, a perpendicular from the center. The ſe

These are equal:
- The rectangle AEB + FEq.
- [f] FBq (FDq.) — [f] 5.2.
- [g] FGq + GDq. — [g] 47.1.
- FGq + [h] GEq + rectangle CED. — [h] 5.2.
- [k] FEq + CED. — [k] 47.1.

[l] Therefore the rectangle AEB = CED. [l] 34.

4 *Case.* If neither of the right lines AB, CD pass through the center, then through the point of intersection E draw the diameter GH. By that which hath been already demonstrated, it appears, that the rectangle AEB = GEH = CED. Which was to be demonstrated.

More easily and generally thus: Joyn AC and ED, then because the angles [a] CEA, DEB, and [b] also C, B (upon the same arch AD) are equal, thence are the triangles CEA, BED [c] equiangular. [d] Wherefore CE. EA :: EB. ED, and [e] consequently CE × ED = AE × EB. Which was to be demonstrated.

[a] 15.1. [b] 21.3. [c] *cor.* 32.1. [d] 4.6. [e] 16.6.

The citations out of the sixth book, both here and in the following proposition, have no dependence upon the same; so that it was free to use them.

PROP. XXXVI.

If any point D be taken without a circle EBC, and from that point two right lines DA, DB fall upon the circle, whereof one DA cuts the circle, the other DB touches it, the rectangle comprehended under the whole line DA that cuts the circle, and under DC that part which is taken from the point given D to the convex of the periphery, shall be equal to the square made of the tangent line.

1 *Case.* If the secant AD pass through the center, then joyn EB; this [a] will make a right angle with the line DB, wherefore DBq + [d] EBq (ECq

[a] 18.3. [d] 3 *ax.*

b 47.1. (ECq) b = EDq c = AD x DC + ECq. There-
c 6.2. fore AD x DC = DEq. Which was to be demonſtrated.

2 *Caſe.* But if AD paſs not through the center, then draw EC, EB, ED, and EF perpendicular to AD: a wherefore AC is biſected in F.

a 3.3.
b 47.1. Becauſe BDq + EBq b = DEq b = EFq +
c 6.2. FDq c = EFq + ADC + FCq d = ADC +
d 47.1. CEq (EBq.) e Therefore is BDq = ADC. Which
e 3 *ax.* was to be demonſtrated.

More eaſily and generally thus: Draw AB
a 32.3. and BC; then becauſe the angles A and DBC a are equal, and the angle D common to both,
b 32.1. thence are the triangles BDC, ADB b equian-
c 4.6. gular. c Wherefore AD. DB :: DB CD, and
d 17.6. d conſequently AD x DC = DBq. Which was to be demonſtrated.

Corollaria.

1. Hence if from any point A taken without a circle, there be ſeveral lines AB, AC drawn, which cut the circle, the rectangles comprehended under the whole lines AB, AC, and the outward parts AE, AF are equal between themſelves.

For if the tangent AD be drawn, then is
a 36.3. CAF = ADq a = BAE.

2. It appears alſo from hence, that if two lines AB, AC drawn from the ſame point do touch a circle, thoſe two lines are equal one to the other.

For if AE be drawn cutting the circle, then is
a 36.3. ABq a = EAF b = ACq.
b 36.3.

3. It is alſo evident, that from a point A taken without a circle there can be drawn but two lines AB, AC, that ſhall touch the circle.

For

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The Third Book of Euclid

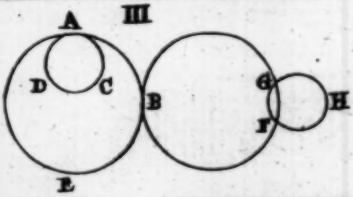
Defin. I

II

XI

XII

III



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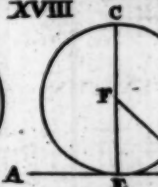

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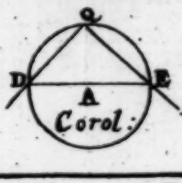

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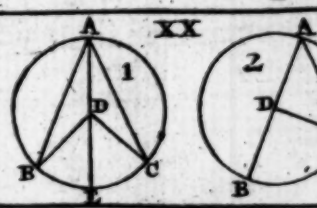

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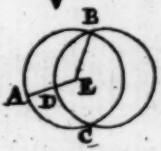

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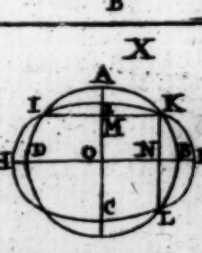

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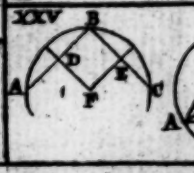

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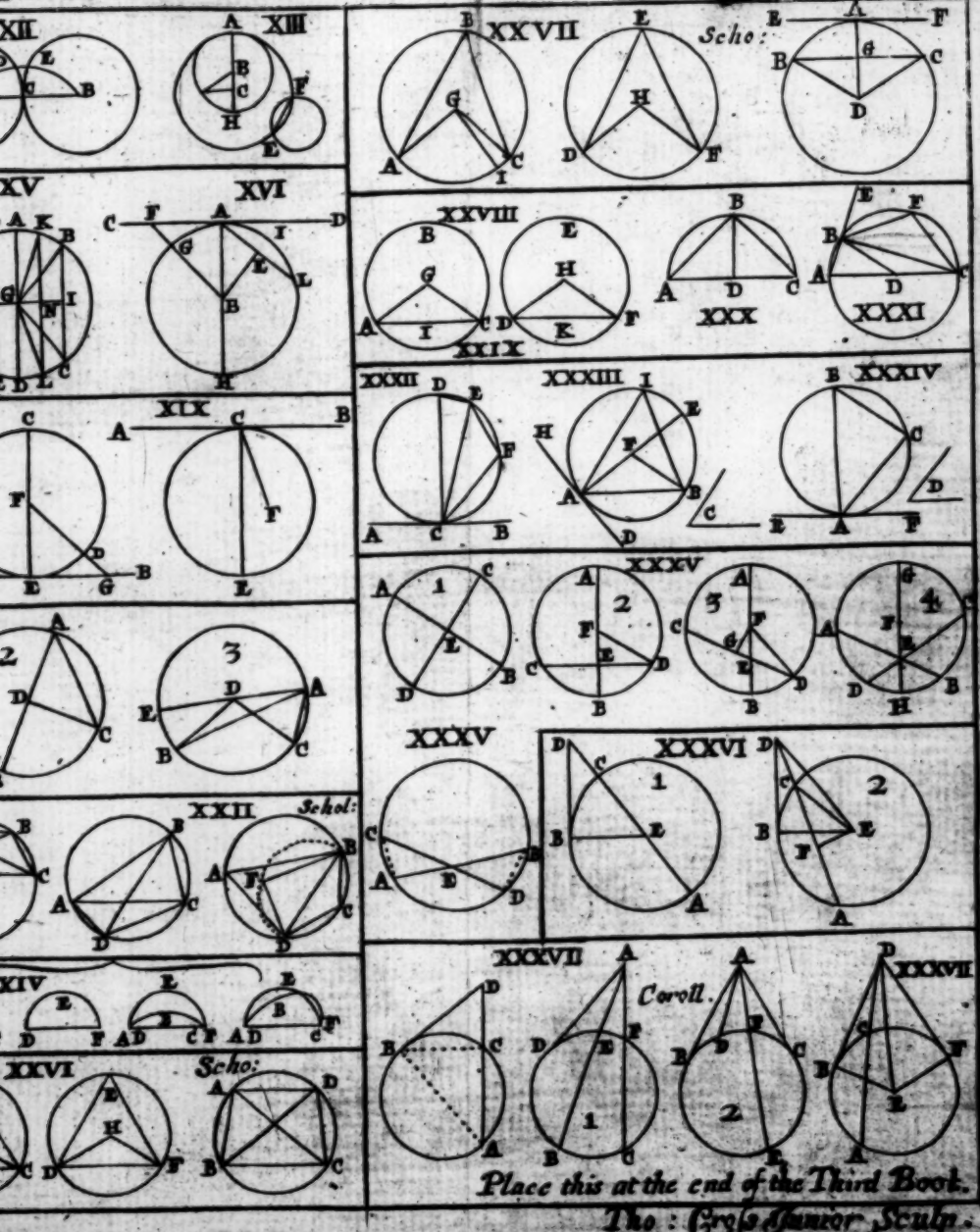
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Coroll.
XXXVII
Place this at the end of the Third Book.
Tho: Cross Junior Sculp.

For if a third line AD be ſaid to touch the circle, thence is AD[c] = AB[c] = AC. [d] Which is abſurd.

[c] 2 *cor.*
[d] 8.3.

4. And on the contrary, it is plain that if two equal right lines AB, AC fall from any point A upon the convex periphery of a circle, and that if one of theſe equal lines AB touch the circle, then the other AC touches the circle alſo.

For if poſſible, let not AC, but another line AD, touch the circle; therefore is AD [e] = AC [f] = AB. [g] Which is abſurd.

[e] 2 *cor.*
[f] *hyp.*
[g] 8.3.

PROP. XXXVII.

If without a circle EBF any point D be taken, and from that point two right lines DA, DB fall on the circle, whereof one line DA cuts the circle, the other DB falls upon it: and if alſo the rectangle comprehended under the whole line that cuts the circle, and under that part of it DC, which is taken betwixt the point D and the convex periphery, be equal to that ſquare which is made of the line DB falling on the circle; I ſay that that line DB ſo falling ſhall touch the circle given.

From the point D [a] let a tangent DF be drawn, and from the center E draw ED, EB, EF. Now becauſe DBq [b] = ADC [c] = DFq, therefore is DB [d] = DF; but EB = EF, and the ſide ED common, [e] therefore the angle EBD = EFD; but EFD is a right angle, and [f] therefore EBD is right alſo, and [g] therefore DB touches the circle. Which was to be demonſtrated.

[a] 17.3.
[b] *hyp.*
[c] 36.3.
[d] 1 *ax.* & *ſch.* 48.1.
[e] 8.1.
[f] 12 *ax.*
[g] *cor.* 16.3.

Corollarium.

From hence it follows, that the [h] angle EDB = EDF.

[h] 8.1.

The End of the Third Book.

THE FOURTH BOOK OF Euclid's *Elements*.

Definitions.

I. A Right lined figure is ſaid to be inſcribed in a right lined figure, when every one of the angles of the inſcribed figure touch every one of the ſides of the figure wherein it is inſcribed.

So the triangle DEF is inſcribed in the triangle ABC.

II. In like manner a figure is ſaid to be deſcribed about a figure, when every one of the ſides of the figure circumſcribed touch every one of the angles of the figure about which it is circumſcribed.

So the triangle ABC is deſcribed about the triangle DEF.

III. A right lined figure is ſaid to be inſcribed in a circle, when all the angles of that figure which is inſcribed, do touch the circumference of the circle.

IV.

IV. A right lined figure is ſaid to be deſcribed about a circle, when all the ſides of the figure which is circumſcribed touch the periphery of the circle.

V. After the like manner a circle is ſaid to be inſcribed in a right lined figure, when the periphery of the circle touches all the ſides of the figure in which it is inſcribed.

VI. A circle is ſaid to be deſcribed about a figure, when the periphery of the circle touches all the angles of the figure which it circumſcribes.

VII. A right line is ſaid to be coapted or applied in a circle, when the extremes thereof fall upon the circumference, as the right line AB.

PROP. I. Probl. 1.

In a circle given ABC, to apply a right line AB equal to a right line given D, which doth not exceed the diameter of the circle.

From the center A by the ſpace AE = D [a] deſcribe a circle meeting with the circle given in B, draw AB. Then is AB [b] = AE [c] = D. Which was to be done.

[a] 3 *poſt.* & 3.1.
[b] 15 *def.* 1.
[c] *conſtr.*

PROP. II. Probl. 2.

In a circle given ABC to inſcribe a triangle ABC, equiangular to a triangle given DEF.

Let the right line GH [a] touch the circle given in A; [b] make the angle HAC = E, [b] and the angle GAB = F: then joyn BC, and the thing is done.

For the angle B [c] = HAC [d] = E, and the angle C [c] = GAB [d] = F: [e] whence alſo the angle BAC = D. Therefore the triangle BAC

[a] 17.3.
[b] 23.1.
[c] 32.3.
[d] *conſtr.*
[e] 32.1.

inſcribed in the circle is equiangular to DEF. Which was to be done.

PROP. III.

About a circle given IABC to deſcribe a triangle LNM equiangular to a triangle given DEF.

Produce the ſide EF on both ſides; at the
[a] 23.1. center I [a] make an angle AIB=DEG, and an angle BIC=DFH. Then in the points A,B,C
[b] 17.3. let three right lines LN,LM,NM [b] touch the circle, and the thing is done.

For it's evident that the right lines LN,LM,
[c] 13 *ax.* MN will meet and make a triangle, [c] becauſe
[d] 18.3. the angles LAI,LBI are right; ſo that the [d] right line AB drawn will make the angles LAB, LBA leſs than two right angles.

[c] *ſch.* 32.1. Since therefore the angle AIB+L [e] = two
[f] 13.1. right angles [f] = DEG+DEF, and AIB [g] =
[g] *conſtr.* DEG: [h] therefore is the angle L=DEF. By
[h] 3 *ax.* the like way of argument the angle M=DFH.
[k] 32.1. [k] Therefore alſo the angle N=D, and therefore the triangle LNM deſcribed about the circle is equiangular to EDF the triangle given. Which was to be done.

PROP. IV.

In a triangle given ABC to inſcribe a circle EFG.

[a] 9.1. [a] Biſect the angles B and C with the right
[b] 12.1. lines BD,CD, meeting in the point D, [b] and draw the perpendiculars DE,DF,DG. A circle deſcribed from the center D through E, will paſs through G and F, and touch the three ſides of the triangle.

[c] *conſtr.* For the angle DBE [c] = DBF, and the angle
[d] 12 *ax.* DEB [d] = DFB, and the ſide DB common.

[e] There-

[e] Therefore DE=DF. By the like argument DG=DF. The circle therefore deſcribed from the center D paſſes through the three points E, F, G; and whereas the angles at E, F, G are right, therefore it touches all the ſides of the triangle. Which was to be done. [e] 26. 1.

Scholium.

Hence the ſides of a triangle being known, their ſegments, which are made by the touchings of the circle inſcribed ſhall be found thus: *Petr. Herig.*

Let AB be 12, AC 18. BC 16; then is AB+BC=28. Out of which ſubduct 18=AC=AE+FC, then remains 10=BE+BF. Therefore BE or BF=5, and conſequently FC or CG =11. Wherefore GA or AE=7.

PROP. V.

About a triangle given ABC to deſcribe a circle FABC,

[a] Biſect any two ſides BA, CA with perpendiculars DF, EF meeting in the point F: I ſay this ſhall be the center of the circle. [a] 10 & 11. 1.

For let the right lines FA, FB, FC be drawn; now becauſe AD [b] = DB and the ſide DF common, and the angles FDA [c] =FDB, therefore is FB [d] = FA. After the ſame manner is FC= FA. Therefore a circle deſcribed about the center F ſhall paſs through the angles of the triangle given, *viz.* B, A, C. Which was to be done. [b] *conſtr.* [c] *conſtr.* & 12 *ax.* [d] 4. 1.

Corollarium.

*31.3. *Hence if a triangle be acute angled, the center ſhall fall within the triangle; if right angled, in the ſide oppoſite to the right angle; and if obtuſe angled, without the triangle.

Scholium.

By the ſame method may a circle be deſcribed, that ſhall paſs through three points given, not being in the ſame ſtrait line.

PROP. VI.

In a circle given EABCD to inſcribe a ſquare ABCD.

[a] 11.1. [a] Draw the diameters AC,BD cutting each other at right angles in the center E. Joyn the extremes of theſe diameters with the right lines AB,BC,CD,DA, and the thing is done.

Now becauſe the four angles at E are right, [b] 26.3. the [b] arches and [c] ſubtended lines AB,BC,CD, [c] 29.3. DA are equal; therefore is the figure ABCD equilateral, and all the angles in ſemicircles, [d] 31.3. and ſo [d] right. [e] Therefore ABCD is a ſquare [e] 29 *def.* 1. inſcribed in a circle given. Which was to be done.

PROP. VII.

About a circle given EABCD to deſcribe a ſquare FHIG.

Draw the diameters AC,BD cutting one the other at right angles; through the extremes of [a] 17.3. theſe diameters [a] draw tangents meeting in F,H,I,G, then I ſay it's done.

For

For becauſe [b] the angles A and C are right, [c] therefore is FG parallel to HI. After the ſame manner is FH parallel to GI, and therefore FHIG is a parallelogram, and alſo right angled. It is equilateral becauſe FG [d] = HI [d] = DB [e] = CA [d] = FH [d] = GI. Wherefore FHIG is a [f] ſquare deſcribed about the circle given. Which was to be done.

[b] 18.3. [c] 28.1. [d] 34.1. [e] 15 *def.* 1. [f] 29 *def.* 1.

Scholium.

A ſquare ABCD deſcribed about a circle is double of the ſquare EFGH inſcribed in the ſame circle.

For the rectangle HB = 2HEF, and HD = 2HGF by the 41.1.

PROP. VIII.

In a ſquare given ABCD to inſcribe a circle IEFGH.

Biſect the ſides of the ſquare in the points H, E, F, G, cutting one the other in I; a circle drawn from the center I through H ſhall be inſcribed in the ſquare.

For becauſe AH and BF are [a] equal and [b] parallel, [c] therefore is AB parallel to HF, parallel to DC. After the ſame manner is AD parallel to EG, parallel to BC. Therefore IA, ID, IB, IC are parallelograms; therefore AH [d] = AE [e] = HI = EI = FI = IG. The circle therefore deſcribed from the center I through H, ſhall paſs through H, E, F, G, and touch the ſides of the ſquare, being the angles H, E, F, G are right. Which was to be done.

[a] 7 *ax.* [b] *ſch.* 34.1. [c] 33.1. [d] 7 *ax.* [e] 34 1.

PROP. IX.

About a ſquare given ABCD to deſcribe a circle EABCD.

Draw the diameters AC, BD cutting one the other in E. About the center E through A deſcribe a circle; then I ſay that circle is deſcribed about the ſquare.

[a] 4 *cor.* 32.1. For the angles ABD and BAC are [a] half of right angles, [b] therefore EA=EB. After the [b] 6.1. ſame manner is EA=ED=EC. The circle therefore deſcribed from the center E paſſes through A,B,C,D, the angles of the ſquare given. Which was to be done.

PROP. X.

To make an Iſoſceles triangle ABD, having each angle at the baſe B, and ADB double to the remaining angle A.

Take any right line AB, and divide it in C. [a] 11.2. [a] ſo that AB × BC may be equal to ACq: from [b] 1.4. the center A through B deſcribe the circle ABD, and in this circle [b] apply BD=AC, and joyn AD: I ſay ABD is the triangle required.

For draw DC, and through the points C,D,A [c] 5.4. [c] draw a circle. Now becauſe AB × BC=ACq, [d] 37.3. [d] it is evident that BD touches the circle ACD, [e] 32.3. which CD cutteth: [e] therefore is the angle [f] 2 *ax.* BDC=A, and therefore the angle BDC+ [g] 32.1. CDA [f] = A+CDA [g] = BCD. But BDC+ [h] 5.1. CDA = BDA [h] = CBD. [k] Therefore the [k] 1 *ax.* angle BCD=CBD, and therefore DC [l] = DB [l] 6.1. = [m] AC; [n] wherefore the angle CDA=A= [m] *conſtr.* BDC, therefore ADB=2A=AED. Which was [n] 5.1. to be done.

This

This conſtruction is analytically found cut thus: Take the thing for done, and let the right line DC biſect the angle BDA : [a] therefore DA. DB : : CA. CB. Alſo becauſe the angle CDA [b] = ½ADB [c] = A, [d] therefore CA = DC. And becauſe the angle DCB [e] = A + CDA = 2A = B, [d] thence will be DB = DC : [f] from whence alſo DB = CA, and ſo DA(BA). CA : : CA. CB, [g] whence BA × CB = CAq.

[a] 3.6.
[b] *conſtr.*
[c] *hyp.*
[d] 6.1.
[e] 32.1.
[f] 2 *ax.*
[g] 17.6.

corollarium.

Whereas all the angles A, B, D [h] make up two right angles, it's evident that A is ⅓ of two right angles.

[h] 32.1.

PROP. XI.

In a circle given ABCDE to deſcribe a pentagone figure ABCDE equilateral and equiangular.

[a] Deſcribe an Iſoſceles triangle FGH, having each angle at the baſe double to the other: [b] inſcribe a triangle CAD equiangular to the ſaid triangle FGH. [c] Biſect the angles at the baſe ACD and ADC, with the right lines DB, CE meeting with the circumference in B and E; joyn the right lines CB, EA, AE, ED, then I ſay it is done.

For it is evident by conſtruction that the angles CAD, CDB, BDA, DCE, ECA are equal: wherefore the [d] arches and [e] ſubtended lines DC, CB, BA, AE, DE are equal. Therefore the pentagone is equilateral and equiangular, [f] becauſe the angles of it BAE, AED, &c. ſtand on equal [g] arches BCDE, ABCD, &c.

[a] 10.4.
[b] 2.4.
[c] 9.1.
[d] 26.3.
[e] 29.3.
[f] 27.3.
[g] 2 *ax.*

A more eaſie practice of this problem ſhall be deliver'd at 10.13.

PROP. IX.

About a ſquare given ABCD to deſcribe a circle EABCD.

Draw the diameters AC, BD cutting one the other in E. About the center E through A deſcribe a circle; then I ſay that circle is deſcribed about the ſquare.

[a] 4 *cor.* 32.1. [b] 6.1. For the angles ABD and BAC are [a] half of right angles, [b] therefore EA=EB. After the ſame manner is EA=ED=EC. The circle therefore deſcribed from the center E paſſes through A,B,C,D, the angles of the ſquare given. Which was to be done.

PROP. X.

To make an Iſoſceles triangle ABD, having each angle at the baſe B, and ADB double to the remaining angle A.

Take any right line AB, and divide it in C. [a] 11.2. [b] 1.4. [a] ſo that AB × BC may be equal to ACq: from the center A through B deſcribe the circle ABD, and in this circle [b] apply BD=AC, and joyn AD: I ſay ABD is the triangle required.

[c] 5.4. [d] 37.3. [e] 32.3. [f] 2 *ax.* [g] 32.1. [h] 5.1. [k] 1 *ax.* [l] 6.1. [m] *conſtr.* [n] 5.1. For draw DC, and through the points C,D,A [c] draw a circle. Now becauſe AB × BC=ACq, [d] it is evident that BD touches the circle ACD, which CD cutteth: [e] therefore is the angle BDC=A, and therefore the angle BDC+CDA [f] = A+CDA [g] = BCD. But BDC+CDA = BDA [h] = CBD. [k] Therefore the angle BCD=CBD, and therefore DC [l] = DB = [m] AC; [n] wherefore the angle CDA=A=BDC, therefore ADB=2A=AED. Which was to be done.

This

This conſtruction is analytically found out thus: Take the thing for done, and let the right line DC biſect the angle BDA : [a] therefore DA. DB : : CA. CB. Alſo becauſe the angle CDA [b] = ½ADB [c] = A, [d] therefore CA = DC. And becauſe the angle DCB [e] = A + CDA = 2A = B, [d] thence will be DB = DC : [f] from whence alſo DB = CA, and ſo DA(BA). CA : : CA. CB, [g] whence BA × CB = CAq.

[a] 3.6.
[b] *conſtr.*
[c] *hyp.*
[d] 6.1.
[e] 32.1.
[f] 2 *ax.*
[g] 17.6.

corollarium.

Whereas all the angles A, B, D [h] make up two right angles, it's evident that A is ⅓ of two right angles.

[h] 32.1.

PROP. XI.

In a circle given ABCDE to deſcribe a pentagone figure ABCDE equilateral and equiangular.

[a] Deſcribe an Iſoſceles triangle FGH, having each angle at the baſe double to the other: [b] inſcribe a triangle CAD equiangular to the ſaid triangle FGH. [c] Biſect the angles at the baſe ACD and ADC, with the right lines DB, CE meeting with the circumference in B and E; joyn the right lines CB, EA, AE, ED, then I ſay it is done.

For it is evident by conſtruction that the angles CAD, CDB, BDA, DCE, ECA are equal: wherefore the [d] arches and [e] ſubtended lines DC, CB, BA, AE, DE are equal. Therefore the pentagone is equilateral and equiangular, [f] becauſe the angles of it BAE, AED, &c. ſtand on equal [g] arches BCDE, ABCD, &c.

[a] 10.4.
[b] 2.4.
[c] 9.1.
[d] 26.3.
[e] 29.3.
[f] 27.3.
[g] 2 *ax.*

A more eaſie practice of this problem ſhall be deliver'd at 10.13.

Corollarium.

Hence each angle of an equilateral and equiangular pentagone is equal to $\frac{3}{5}$ of two right angles, or $\frac{6}{5}$ of one right angle.

Scholium.

Petr. Herig. *Generally all figures of odd number of sides are inscribed in circles by the help of Isosceles triangles, whose angles at the base are multiplices of those at the top: and figures of even number of sides are inscribed in a circle by the help of Isosceles triangles, whose angles at the base are multiplices sesquialter of those at the top.*

As in the Isosceles triangle CAB, if the angle A=3C=B, then will AB be the side of a heptagone. If A=4C, then is AB the side of an eneagone. But if A=$\frac{1}{2}$C, then is AB the side of a square. And if A=2$\frac{1}{2}$C, AB will subtend the sixth part of a circumference: and likewise if A=3$\frac{1}{2}$C, then will AB be the side of an octagone.

P R O P. XII.

About a circle given FABCDE, to describe an equilateral and an equiangular pentagone HIKLG.

a 11.4. a Inscribe a pentagone ABCDE in the circle given, and from the center draw the right lines FA, FB, FC, FD, FE; and to those lines draw so many perpendiculars GAH, HBI, IGK, KDL, LEG, meeting in the points H, I, K, L, G: then I say it is done. For because GA, GE from the same point G b touch the circle, c therefore is GA=GE, and d therefore the angle GFA=

b *cor.* 16.3.
c 2 *cor.* 36.3.
d 8.1.

GFE,

GFE, therefore the angle AFE = 2GFA. After the ſame manner is the angle AFH = FHB, and conſequently the angle AFB = 2AFH.
[e] But the angle AFE = AFB, [f] therefore the angle GFA = AFH: but alſo the angle FAH = FAG: and the ſide FA is common; [h] therefore HA = AG = GE = EL, &c. [k] Therefore HG, GL, LK, KI, IH the ſides of the pentagone are equal, the angles alſo are, becauſe double of the [l] equal angles AGF, AHF. therefore, &c.

[e] 27.3. [f] 7 *ax.* [g] 12 *ax.* [h] 26.1. [k] 2 *ax.* [l] 32.1.

Corollarium.

After the ſame manner if any equilateral and equiangled figure be deſcribed in a circle, and at the extreme points of the ſemidiameters drawn from the center at angles, be drawn perpendicular lines to the ſaid diameters; I ſay that theſe perpendiculars ſhall make another figure of as many equal ſides and equal angles, deſcribed about the circle.

PROP. XIII.

In an equilateral and equiangular pentagone given ABCDE, to inſcribe a circle FGHK.

[a] Biſect two angles of the pentagone A and B, with the right lines AF, BF meeting in the point F: from F draw the perpendiculars FG, FH, FI, FK, FL. Then a circle deſcribed about the center F through G, will touch all the ſides of the pentagone.

[a] 9.1.

Draw FC, FD, FE; becauſe BA [b] = BC, and the ſide BF common, and the angle FBA [c] = FBC, [d] therefore is AF = FC, and the angle FAB = FCB; but the angle FAB [e] = ½BAE = ½BCD. Therefore the angle FCB = ½BCD.

[b] *hyp.* [c] *conſtr.* [d] 4.1. [e] *hyp.*

After the ſame manner are all the whole angles C,D,E biſected. Now whereas the angle FGB [f] = FHB, and the angle FBH = FBG, and the ſide FB is common; [g] therefore is FG = FH. In like manner are all the right lines FH, FI, FK, FL, FG equal. Therefore a circle deſcribed about the center F through G, paſſes through the points H,I,K,L, and [h] touches the ſides of the pentagone, becauſe the angles at thoſe points are right. Which was to be done.

f 12 *ax.*
g 26.1.
h *cor.* 16.3.

Corollarium.

Hence if any two neareſt angles of an equilateral and equiangular figure, and from that point in which the lines meet that biſect the angles, be drawn right lines to the remaining angles of the figure, all the angles of the figure ſhall be biſected.

Scholium.

By the ſame method ſhall a circle be inſcribed in any equilateral and equiangular figure.

PROP. XIV.

About a pentagone given ABCDE equilateral and equiangular, to deſcribe a circle FABCDE.

Biſect any two angles of the pentagone with the right lines AF, EF meeting in the point F, the circle deſcribed from the center F through A ſhall be deſcribed about the pentagone.

For let FC, FD, FE be drawn, [a] then the angles C,D,E are biſected; [b] and therefore FA, FB, FC, FD, FE are equal, therefore the circle deſcribed about the center F paſſes through A,B,

a *cor.* 13.4.
b 6.1.

A,B,C,D,E all the angles of the pentagone. Which was to be done.

Scholium.

By the same art is a circle described about any figure which is equilateral and equiangular.

PROP. XV.

In a circle given GABCDEF to inscribe an hexagone (or six-sided figure) equilateral and equiangular ABCDEF.

Draw the diameter AD; about the center D through the center G describe a circle, cutting the circle given in the points C and E. Draw the diameters CF,EB, and joyn AB,BC, CD,DE,EF,FA, then I say it's done.

For the angle CGD [a] = $\frac{1}{3}$ of two right = DGE [b] = AGF [b] = AGB. [c] Therefore BGC = $\frac{1}{3}$ of two right = FGE, therefore the [d] arches and [e] subtenses AB,BC,CD, DE, EF are equal. Therefore the hexagone is equilateral, but it is equiangled also, [f] because all the angles of it stand upon equal arches.

[a] 32.1. [b] 15.1. [c] *Cor.* 13.1. [d] 26.3. [e] 29.3. [f] 27.3.

Corollaria.

1. Hence the side of an hexagone inscribed in a circle is equal to the semidiameter.

2. Hereby an equilateral triangle ACE may very easily be described in a circle given.

Scholium.

To make a true hexagone upon a right line given CD. *Andr. Tacq.*

[a] Make

[a] 1.1. [a] Make an equilateral triangle CGD upon the line given CD, about the center G through C and D describe a circle, that circle shall contain the hexagone made upon the given line CD.

PROP. XVI.

In a circle given AEBC to inscribe a quindecagone (or fifteen sided figure) equilateral and equiangular.

[a] 11.4. [a] Inscribe an equilateral pentagone AEFGH
[b] 3.4. in the circle given, and [b] also an equilateral triangle AEC. then I say BF is the side of the quindecagone required.

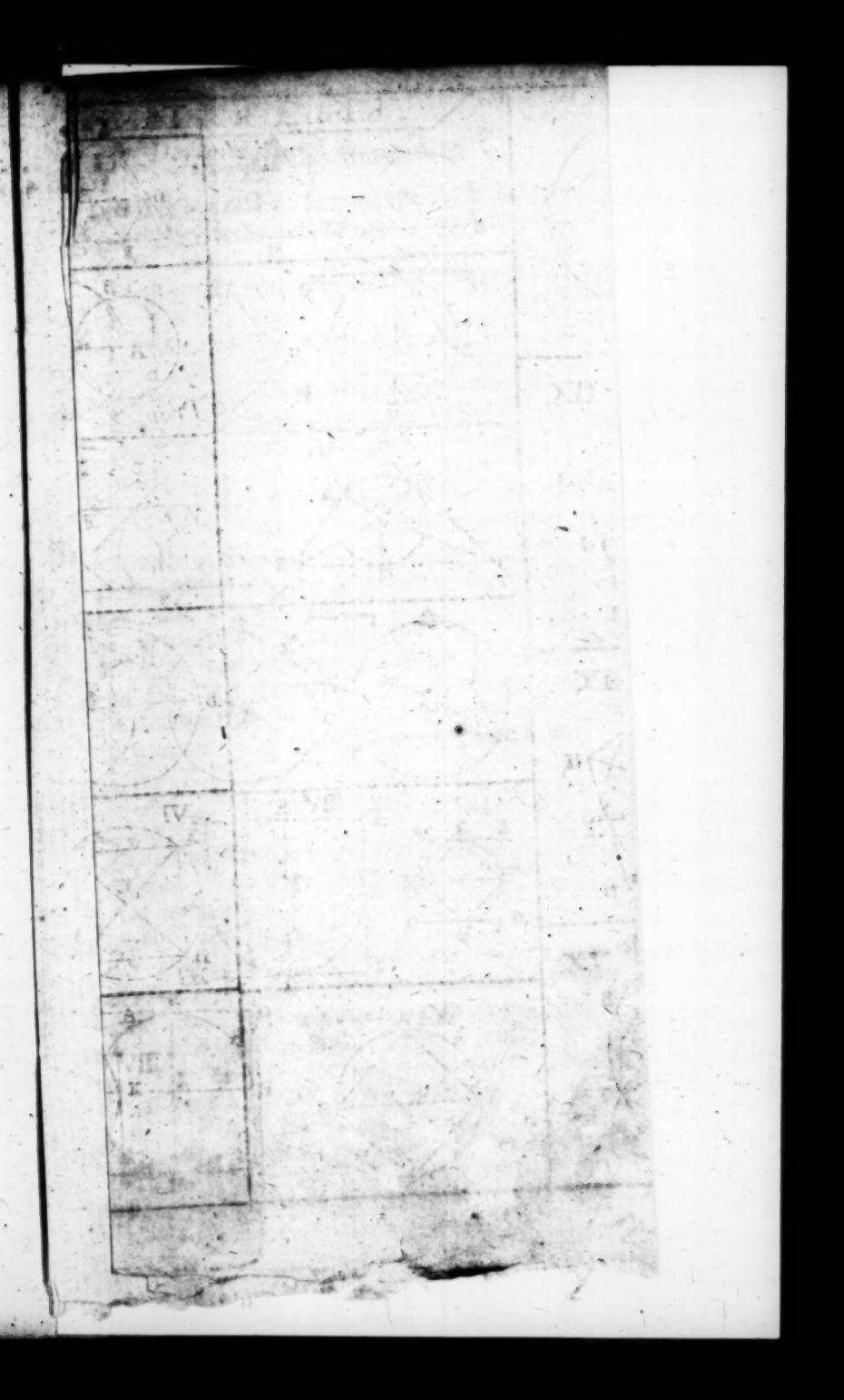
[c] *constr.* For the arch AB [c] is $\frac{1}{5}$ or $\frac{3}{15}$ of that periphery, whereof AF is $\frac{1}{3}$ or $\frac{5}{15}$: therefore the remaining part BF is $\frac{2}{15}$ of the periphery. And therefore the quindecagone, whose side is BF,
[d] 27.3. is equilateral; but it is equiangular also, [d] because all the angles insist on equal arches of a circle, whereof every one $\frac{2}{15}$ of the whole circumference. Therefore, &c.

Scholium.

A circle is geometrically divided into parts,
- 4,8,16, &c. by 6,4 and 9,1.
- 3,6,12, &c. by 15,4 and 9,1.
- 5,10,20, &c. by 11,4 and 9,1.
- 15,30,60, &c. by 16,4 and 9,1.

Any other way of dividing the circumference into any parts given is as yet unknown; wherefore in the construction of ordinate figures we are forced to have recourse to mechanick artifices, concerning which you may consult the Writers of Practical Geometry.

The End of the Fourth Book.



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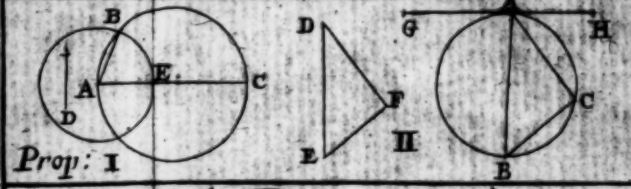
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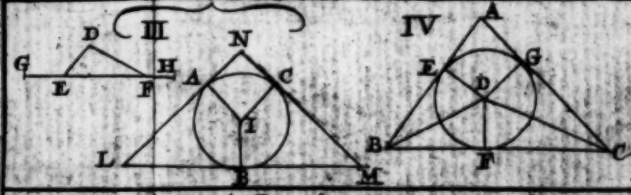
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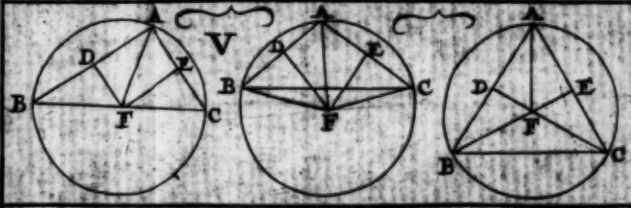
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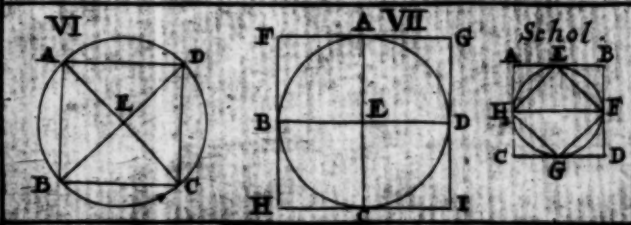
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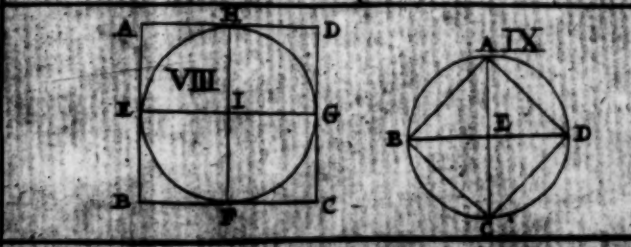
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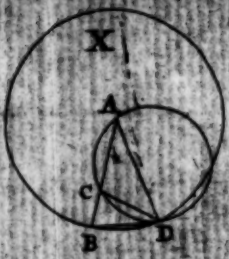
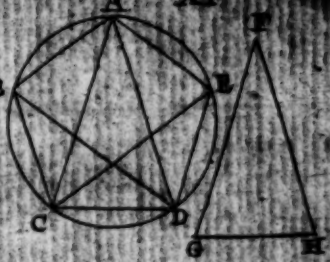


XI



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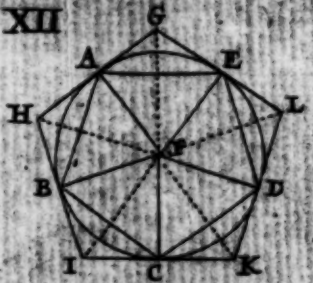
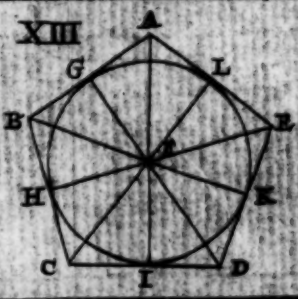


Schol.

XIII



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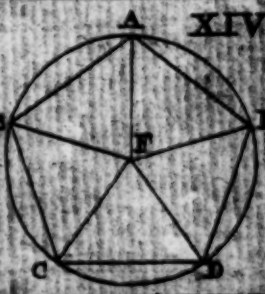
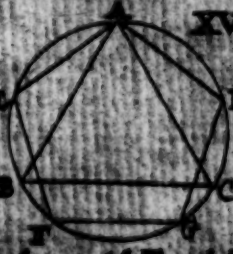


XV

XVI



Place this at the end of the Fourth Book

THE FIFTH BOOK OF Euclid's *Elements*.

Definitions.

I. A Part is a magnitude of a magnitude, a leſs of a greater, when the leſs meaſureth the greater.

II. Multiplex is a greater magnitude in reſpect of a leſſer, when the leſſer meaſureth the greater.

III. Ratio (or rate) is the mutual habitude or reſpect of two magnitudes of the ſame kind each to other, according to quantity.

In every ratio that quantity which is referred to another quantity, is called the antecedent of the ratio; and that to which the other is referred, is called the conſequent of the ratio. As in the ratio of 6 to 4, 6 is the antecedent, and 4 the conſequent.

Note. *The quantity of any ratio is known by dividing the antecedent by the conſequent. As the ratio of 12 to 5 is expreſſed by $\frac{12}{5}$: or the quantity of the ratio of A to B is*

A

$\frac{A}{B}$. *Wherefore often for brevity ſake we denote the quantities of rations thus:* $\frac{A}{B} \sqsubset$, *or* $=$, *or* $\sqsupset \frac{C}{D}$. *That is, the ratio of A to B is greater, equal, or leſs than the ratio of C to D. And this Note muſt be diligently obſerved in the underſtanding of the following Book.*

Concerning the diverſe ſpecies of ratio's you may pleaſe to conſult interpreters.

IV. Proportion is a ſimilitude of ratio's.

That which is here termed proportion, *is more rightly called* proportionality *or* analogy: *for proportion commonly denotes no more than the ratio betwixt two magnitudes.*

V. Thoſe numbers are ſaid to have a ratio betwixt them, which being multiplied may exceed one the other,

E.12	A.4.	B.6	G.24
F.30	C.10.	D.15	H.60

VI. Magnitudes are ſaid to be in the ſame ratio, the firſt A to the ſecond B, and the third C to the fourth D, when the equimultiplices E and F of the firſt A, and the third C compared with the equimultiplices G,H of the ſecond B and the fourth D, according to any multiplication whatſoever, either both together E,F are leſs than G,H; or both together equal to them, or both together exceed them, if thoſe be taken E,G and F,H, which anſwer one to the other.

The note hereof is:: as A.B::C.D: that is, as A is to B, ſo is C to D; which ſignifies that A to B and C to D are in the ſame ratio. We ſome-

ſometimes thus expreſs it, $\frac{A}{B} = \frac{C}{D}$ *: That is,* $A.B :: C.D$.

VII. Magnitudes that have the ſame ratio (A.B :: C.D) are called proportional.

E.30	A.6. B.4	G.28
F.60	C.12.D.9	H.63

VIII. When of equimultiplices, E the multiplex of the firſt magnitude A, exceeds the G the multiplex of the ſecond B; but F the multiplex of the third C exceeds not H the multiplex of the fourth D: then the firſt A to the ſecond B has a greater ratio than the third C to the fourth D.

If $\frac{A}{B} \sqsubset \frac{C}{D}$, *it is not neceſſary from this definition that E ſhould always exceed G, when F is leſs than H: but it is granted that this may be.*

IX. Proportionality conſiſts in three terms at leaſt, whereof the ſecond ſupplies the place of two.

X. When three magnitudes A,B,C are proportional, the firſt A is ſaid to have a duplicate ratio to the third C, of that it hath to the ſecond B. But when four magnitudes A,B,C,D are proportional, the firſt A is ſaid to have a triplicate ratio to the fourth D, of what it had to the ſecond B; and ſo always in order one more, as the proportion ſhall be extended.

Duplicate ratio is thus expreſſed, $\frac{A}{C} = \frac{A}{B}$ *twice, that is, the ratio of A to C is double of the ratio of A to B. Triple ratio is thus expreſſed,* $\frac{A}{D} = \frac{A}{B}$ *thrice; that is, the ratio of A to D is triple of the ratio of A to B.*

∴

$\div\div$ *denotes continued proportionals, as A,B,C, D; or* 2,6,18,64 *are* $\div\div$.

XI. Magnitudes homologous or like in ratio are antecedents to antecedents, and conſequents to conſequents.

As if A.B::C.D; then A and C, and B and D, are homologous or magnitudes like in ratio.

XII. Alternate proportion is the comparing of antecedent to antecedent, and conſequent to conſequent.

As if A.B::C.D; therefore alternately or by permutation A.C::B.D by the 16 *of* 5.

In this definition, and the five following, names are given to the ſix ways of arguing, which are often uſed by Mathematicians: the force of which inferences depends on the propoſitions of this book, which are named in their explications.

XIII. Inverſe ratio is when the conſequent is taken as the antecedent, and ſo compared to the antecedent as the conſequent.

As A.B::C.D; therefore inverſly B.A::D.C, by cor. 4.5.

XIV. Compounded ratio is when the antecedent and conſequent taken both as one, are compared to the conſequent it ſelf.

As A.B::C.D; therefore by compoſition A+B.B::C+D.D, by 18.5.

XV. Divided ratio is when the exceſs wherein the antecedent exceedeth the conſequent, is compared to the conſequent.

As A.B::C.D; therefore by diviſion A—B.B::C—D.D, by 17.5.

XVI. Converſe ratio is when the antecedent is compared to the exceſs, wherein the antecedent exceeds the conſequent.

As

As A.B : : C.D ; therefore by converſe ratio A.A—B : : C.C—D, by the coroll. of the 19 *of the* 5.

XVII. Proportion by equality is where there are taken more magnitudes than two in one order, and alſo as many magnitudes in another order, comparing two to two being in the ſame ratio; it cometh to paſs that as in the firſt order of magnitudes the firſt is to the laſt, ſo in the ſecond order of magnitudes is the firſt to the laſt. Or otherwiſe it is a compariſon of the extremes together, the mean magnitudes being taken away. *See prop.* 22. & 23. *of the* 5.

XVIII. Ordinate proportionality is, when as the antecedent is to the conſequent, ſo is the antecedent to the conſequent; and as the conſequent is to any other, ſo is the conſequent to any other.

As A.B : : D.E, alſo B.C : : E.F ; it ſhall be true alſo A.C : : D.F, by the 22 *of the* 5.

XIX. Inordinate proportion is when three magnitudes being put, and others alſo which are equal to theſe in multitude; as in the firſt magnitudes the antecedent is to the conſequent, ſo in the ſecond magnitudes is the antecedent to the conſequent. And as in the firſt magnitudes the conſequent is to any other, ſo in the ſecond magnitudes any other thing to the antecedent.

As A.B : : F.G, alſo B.C : : E.F ; it ſhall be true in inordinate proportion A.C : : E.G, by the 23 *of the* 5.

XX. Any number of magnitudes being put, the proportion of the firſt to the laſt is compounded out of the proportions of the firſt to the ſecond, the ſecond to the third, and the third to the fourth, and ſo forwards till the proportion ariſe.

Let

Let there be any number of magnitudes A, B, C, D, by this definition $\frac{A}{D} = \frac{A}{B} + \frac{B}{C} + \frac{C}{D}$.

Axiom.

Magnitudes equimultiplices to the ſame multiplex, are alſo equimultiplices betwixt themſelves.

PROP. I.

If there be a number of magnitudes how many ſoever AB, CD, equimultiplices to a like number of magnitudes E, F each to other; how multiplex one magnitude AB is to one E, ſo multiplices are all the magnitudes AB + CD to all the other magnitudes E + F.

Let AG, GH, HB the parts of the quantity AB be equal to E, and alſo let CI, IK, KD the parts of the quantity CD be equal to F. The number of theſe are put equal to thoſe. Now whereas AG + CI [a] = E + F, [a] and GH + IK = E + F, [a] and HB + KD = E + F; it is evident that AB + CD doth ſo often contain E + F. as one AB contains E. Which was to be demonſtrated.

[a] 2 *ax.*

PROP. II.

If the firſt AB be equimultiplex to the ſecond C, as the third DE is to the fourth F; and if the fifth BG be equimultiplex to the ſecond C, as the ſixth EH is to the fourth F: then ſhall the firſt compounded with the fifth (AG) be equimultiplex to the ſecond C, as the third compounded with the ſixth (DH) is to the fourth F.

The

The number of parts in AB equal each to C, is put equal to the number of parts in DE, whereof each part is equal to F. Likewiſe the number of parts in BG is put equal to the number of parts in EH. Therefore the number of pars in AB+BG, is equal to the number of parts in DE+EH. [a] That is, the whole line AG is as equimultiplex of C, as the whole line DH is of F. Which was to be demonſtrated. [a] 2 *ax.*

PROP. III.

If the firſt A be equimultiplex of the ſecond B, and the third C of the fourth D, and there be taken EI, FM equimultiplices of the firſt and third, then will each of the magnitudes taken be alike equimultiplex of both, the one EI to the ſecond B, the other FM to the fourth D.

Let EG, GH, HI the parts of the multiplex EI be equal to A: alſo let FK, KL, LM the parts of the multiplex FM be equal to F, [a] the number of theſe is equal to the number of thoſe. Moreover A (that is) EG, or GH, or HI, is put as multiplex of B, as C or FK, &c. of D. [b] Therefore EG+GH is equimultiplex of the ſecond B, as FK+KL is of the fourth D. [c] By the ſame way of argument is EI (EH+HI) as multiplex of B, as FM (FL+LN) is of D. Which was to be demonſtrated. [a] *byp.* [b] 2.5. [c] 2.5.

PROP. IV.

If the firſt A have the ſame ratio to the ſecond B, as the third C to the fourth D; then alſo E and F, the equimultiplices of the firſt A and the third C, ſhall have the ſame ratio to G and H the equimultiplices of the ſecond B and the fourth D, according to any multiplication,

tiplication, if ſo taken as they anſwer each to other (E.G :: F.H.)

Take I and K the equimultiplices of E and F, and alſo L and M, the equimultiplices of G and H. [a] Then is I as multiplex of A, as K of C;

[a] 3.5. [a] and alſo L is as multiplex of B, as M of D.

[b] hyp. Therefore whereas it is A.B [b] :: C.D, according to the ſixth definition; if I be ⊏, =, ⊐ L then conſequently after the ſame manner is K ⊏, =, ⊐ M. Therefore when I and K are taken as multiplices of E and F, as L and M of G and H, then will it be by the ſeventh definition E. G :: F. H. Which was to be demonſtrated.

Corollarium.

From hence is wont to be demonſtrated the proof of inverſe ratio.

For becauſe A.B :: C.D, therefore if E ⊏, [c] 6 def. 5. =, ⊐ G, then is [c] likewiſe F ⊏, =, ⊐ H. Therefore it is evident that if G ⊏, =, ⊐ E, [d] 6 def. 5. then is H ⊏, =, ⊐ F : [d] therefore B.A :: D.C. Which was to be demonſtrated.

PROP. V.

If a magnitude AB be as multiplex of a magnitude CD, as a part taken from the one AE of a part taken from the other CF; the reſidue of the one EB ſhall be as multiplex of the reſidue of the other FD, as the whole AB is of the whole CD.

Take any other GA, which ſhall be as multiplex of FD the reſidue, as AB is of the whole CD, or as the part taken away AB is of the part [a] 1.5. taken away CF. [a] Therefore the whole GA + AE is as multiplex of the whole CF + FD, as the

the one AE of the one CF, that is, as AB of CD. Therefore GE [b] = AB, and [c] ſo AE that was common being taken away, there remains GA = EB. Therefore, &c.

[b] 6 *ax.*
[c] 3 *ax.*

PROP. VI.

If two magnitudes AB, CD be equimultiplices of two magnitudes E, F, and ſome magnitudes AG and CH equimultiplices of the ſame E, F be taken away, then the reſidues GB, HD are either equal to theſe magnitudes E, F, or elſe equimultiplices of them.

For becauſe the number of parts in AB, whereof each is equal to E, is put equal to the number of parts in CD, whereof each is equal to F; and alſo the number of parts in AG, equal to the number of parts in CH. If from one you take AG, and from the other CH, [a] then remains the number of parts in the remainder GB, equal to the number of parts in HD. Therefore if GB be once E, then is HD once F; if GB be many times E, then is HD ſo of F. Which was to be demonſtrated.

[a] 3 *ax.*

PROP. VII.

Equal magnitudes A and B have to the ſame magnitude C the ſame proportion or ratio. And one and the ſame magnitude C hath the ſame ratio to equal magnitudes A and B.

Take D and E equimultiplices of the equal magnitudes A and B, and F any wiſe multiplex of C, then is D [a] = E. Wherefore if D ⊏, =, ⊐ F, then alſo E will be ⊏, =, ⊐ F. [b] Therefore A. C :: B. C, and [c] by inverſion C. A :: C. B. Which was to be demonſtrated.

[a] 6 *ax.*
[b] 6 *def.* 5.
[c] *cor.* 4. 5.

Scho-

Scholium.

F If inſtead of the multiplex F two equimultiplices be taken, it ſhall be the ſame way proved, that equal magnitudes have the ſame ratio to other magnitudes, that are equal between themſelves.

PROP. VIII.

Of unequal magnitudes AB, AC, the greater AB hath a greater ratio to the ſame third line D, than the leſſer AC; and the ſame third line D hath a greater ratio to the leſſer AC, than to the greater AB.

Take EF, EG equimultiplices of the ſaid AB, AC, ſo that EH being multiplex of D be greater than EG, but leſſer than EF, (which will eaſily happen if both EG and GF be taken greater than D.) It is manifeſt from 8 def. 5. that $\frac{AB}{D} \sqsupset \frac{AC}{D}$, and $\frac{D}{AB} \sqsubset \frac{D}{AC}$. Which was to be demonſtrated.

PROP. IX.

Magnitudes which to one and the ſame magnitude have the ſame ratio, are equal the one to the other. And if a magnitude have the ſame ratio to other magnitudes, thoſe magnitudes are equal one to the other.

1 *Hypotheſis.* If A.C :: B.C, I ſay that A = B.

a 8. 5. For let A be greater or leſs than C, a then is $\frac{A}{C} \sqsupset$ or $\sqsubset \frac{B}{C}$. Which is contrary to the Hypotheſis.

2 *Hy-*

2 *Hypotheſis.* If C.B :: C.A, I ſay that A = B.

For let A be ⊏ B, [b] then $\frac{C}{B} \sqsubset \frac{C}{A}$. Which is [b] 8.5. againſt the Hypotheſis.

PROP. X.

Of magnitudes having ratio to the ſame magnitude, that which has the greater ratio is the greater magnitude: and that magnitude to which the ſame carries a greater ratio, is the leſſer magnitude.

1 *Hypotheſis.* If $\frac{A}{C} \sqsubset \frac{B}{C}$, I ſay that A ⊏ B.

For if it be ſaid that A = B, [a] then A.C :: B.C. [a] 7.5. Which is contrary to the Hypotheſis. If A ⊐ B, [b] then is $\frac{A}{C} \sqsupset \frac{B}{C}$. Which is alſo againſt the [b] 8.5. Hypotheſis.

2 *Hypotheſis.* If $\frac{C}{B} \sqsubset \frac{C}{A}$, I ſay that B ⊐ A.

For if you ſay B = A, it's againſt the Hypotheſis; for it will [c] follow that C.B :: C.A. If you [c] 7.5. ſay B ⊏ A, [d] then is $\frac{C}{A} \sqsubset \frac{C}{B}$. Which is alſo [d] 8.5. againſt the Hypotheſis.

PROP. XI.

Proportions which are one and the ſame to any third, are alſo the ſame one to another.

Let A.B :: E.F, and C.D :: E.F, I ſay that A.B :: C.D. Take G, H, I, the equimultiplices of A, C, E, and K, L, M, the equimultiplices of B, D, F. Now [a] becauſe A.B :: E.F, if G ⊏, =, ⊐ K, [b] then after the ſame manner I ⊏, =, ⊐ M. And likewiſe [a] becauſe E.F :: C.D, if I ⊏, [a] *hyp.* [b] 6 *def.* 5.

I ⊏,=,⊐ M, [b] then is H likewiſe ⊏,=, ⊐ L. [c] Wherefore A.B :: C.D. Which was to be demonſtrated.

[c] 6 *def.* 5.

Scholium.

Proportions that are one and the ſame to the ſame proportions, are the ſame betwixt themſelves.

PROP. XII.

If any number of magnitudes A, B, C, D, E, and F be proportional, as one of the antecedents A is to one of the conſequents B, ſo are all the antecedents A, C, E to all the conſequents B, D, F.

Take the equimultiplices of the antecedents G, H, I, and of the conſequents K, L, M; becauſe that as multiplex as one G is of one A, [a] ſo multiplices are all G, H, I of all A, C, E; and likewiſe as multiplex as one K is of one B, ſo multiplices are all K, L, M of all B, D, F. Moreover, becauſe A.B [b] :: C.D [b] :: E.F, if G be ⊏,=,or ⊐ K, then will H likewiſe be ⊏,=, ⊐ L, and I ⊏,=, ⊐ M; and therefore if G ⊏,=, ⊐ K, in like manner will C+H+I be ⊏,=, ⊐ K +L+M. [c] Wherefore A. B :: A+C+E.B +D+F. Which was to be demonſtrated.

[a] 1.5.

[b] *hyp.*

[c] 6 *def.* 5.

Corollarium.

From hence if like proportionals be added to like proportionals, the wholes ſhall be proportional.

PROP.

PROP. XIII.

If the first A have the same ratio to the second B, that the third C hath to the fourth D; and if the third C have a greater proportion to the fourth D, than the fifth E to the sixth F; then also shall the first A have a greater proportion to the second B, than the fifth E to the sixth F.

Take G,H,I equimultiplices of A,C,E. and K,L,M equimultiplices of B,D,F. Now because that A. B :: C. D, if H ⌐ L, [a] then is G ⌐ K; [a] 6 *def.* 5.

but because $\frac{C}{D} \sqsubset \frac{E}{F}$, [b] it may be that H ⌐ L, [b] 8 *def.* 5.

and yet I not ⌐ M. [c] Therefore $\frac{A}{B} \sqsubset \frac{E}{F}$. [c] 8 *def.* 5.

Which was to be demonstrated.

Scholium.

But if $\frac{C}{D} \sqsupset \frac{E}{F}$, then also is $\frac{A}{B} \sqsupset \frac{E}{F}$. Also if $\frac{A}{B} \sqsubset \frac{C}{D} \sqsubset \frac{E}{F}$, then is $\frac{A}{B} \sqsubset \frac{E}{F}$. And if $\frac{A}{B} \sqsupset \frac{C}{D} \sqsupset \frac{E}{F}$, then is $\frac{A}{B} \sqsupset \frac{E}{F}$.

PROP. XIV.

If the first A have the same ratio to the second B, that the third C hath to the fourth D; and if the first A be greater than the third C, then shall the second B be greater than the fourth D. But if the first A be equal to the third C, then the second B shall be equal to the fourth D. But if A be lesser, then is B also lesser.

[a] 8.5. [b] *hyp.* [c] 13.5. [d] 10.5. [e] 7.5. [f] *hyp.* [g] 11.5. & 9.5.

Let A ⊏ C, [a] then $\frac{A}{B} \sqsubset \frac{C}{B}$: [b] but $\frac{A}{B} = \frac{C}{D}$; [c] therefore $\frac{C}{D} \sqsubset \frac{C}{B}$, [c] therefore B ⊏ D. By the like way of argument if A ⊐ C, [d] then is B ⊐ D. But if A be put equal to C, then C.B :: [e] A.B [f] :: C.D, [g] therefore B = D. Which was to be demonstrated.

Scholium.

By an argument *à fortiori*, if $\frac{A}{B} \sqsupset \frac{C}{D}$, and A ⊏ C, then is B ⊏ D. Likewise if A = B, then is C = D; and if A ⊏ or ⊐ B, then also is C ⊏ or ⊐ D.

PROP. XV.

Parts C and F are in the same ratio with their like multiplices AB and DE, if taken correspondently. (AB. DE :: C.F.)

Let AG, GB parts of the multiplex AB be equal to C, and let DH, HE parts of the multiplex DE be equal to F. [a] The number of these parts is equal to the number of those; therefore whereas [b] AG.DH :: C.F :: GB.HE, therefore is [c] AG + GB (AB). DH + HE (DE) :: C.F. Which was to be demonstrated.

[a] *hyp.* [b] 7.5. [c] 12.5.

PROP. XVI.

If four magnitudes A, B, C, D be proportional, they also shall be alternately proportional. (A.C :: B.D.)

Take E and F equimultiplices of A and B; take also G and H equimultiplices of C and D. Therefore E.F [a] :: A. B [b] :: C. D [a] :: G. H. Where-

[a] 15.5. [b] *hyp.*

Wherefore if E⊏,=,⊐G, [c] then likewiſe is F⊏,=,⊐H. [d] Therefore A.C::B.D. Which was to be demonſtrated.

[c] 11.5. & 14.5.
[d] 6 *def.*5.

Scholium.

Alternate ratio has place onely then, when the quantities are of the ſame kind: for heterogeneous quantities are not compared together.

PROP. XVII.

If magnitudes compounded be proportional (AB. CB :: DE. FE.) they ſhall be proportional alſo when divided. (AC. CB :: DF. FE.)

Take GH,HL,IK,KM in order, the equimultiplices of AC,CB,DF,FE; and alſo LN,MO the equimultiplices of CB,FE. The whole GL is [a] as multiplex of the whole AB, as one GH of one AC, [b] that is, as IK of DF, [c] or as the whole IM of the whole DE. Alſo HN (HL+LN) is as [d] multiplex of CB, as KO (KM+MO) is of FE. Therefore whereas by Hypotheſis AB.BC::DE.EF, if GL be ⊏,=,⊐HN, then likewiſe [e] will IM ⊏,=,⊐KO. Take from theſe HL,KM that are common, and if the remainder GH be ⊏,=,⊐LN, [f] then will IK ⊏,=,⊐MO. [g] Whence AC.CB::DF.FE. Which was to be demonſtrated.

[a] 1.5.
[b] *conſtr.*
[c] 1.5.
[d] 2.5.
[e] 6 *def.*5.
[f] 5 *ax.*
[g] 6 *def.*5.

PROP. XVIII.

If magnitudes divided be proportional (AB. BC :: DE. EF) the ſame alſo being compounded ſhall be proportional. (AC.CB:: DF.FE.)

For if it can be let AC.CB:: DF.FG⊐FE; [a] then by diviſion will AB.BC:: DG.GF, [b] that

[a] 17.5.
[b] *hyp.* & 11.5.

is, DG.GF :: DE.EF ; and being DG ⊏ DE, [c] therefore is GF ⊏ EF. [d] Which is absurd. The like absurdity will follow if it be said AB. CB :: DE.GF ⊏ FE.

[c] 14.5.
[d] 9 *ax.*

PROP. XIX.

If the whole AB be to the whole DE, as the part taken away AC is to the part taken away DF ; then shall the residue CB be to the residue FE, as the whole AB is to the whole DE.

Because [a] AB.DE :: AC.DF, [b] therefore by permutation AB.AC :: DE.DF, [c] and thence by division AC.CB :: DF.FE. [b] Wherefore again by permutation AC.DF :: CB.FE, [d] that is, AB.DE :: CB.FE. Which was to be demonstrated.

[a] *hyp.*
[b] 16.5.
[c] 17.5.
[d] *hyp.* & 11.5.

Corollaria.

1. Hence if like proportionals be subtracted from like proportionals, the residue shall be proportional.

2. *Hence is converse ratio demonstrated.*

Let AB.CB :: DE.FE, I say that AB.AC :: DE.DF ; for by [a] permutation AB.DE :: CB. FE, [b] therefore AB. DE :: AC. DE, whence again by permutation AB.AC :: DE.DF. Which was to be demonstrated.

[a] 16.5.
[b] 19.5.

PROP. XX.

If there be three magnitudes A, B, C, and others D, E, F, equal to those in number ; which being taken two and two in each order are in the same ratio, (A.B :: D.E, and B.C :: E.F) and if of equality the first A be greater than the third C, then shall the fourth D

be greater than the ſixth F. But if the firſt A be equal to the third C, then the fourth D is ſo to the ſixth F; and if A be leſs than C, ſo D is leſs than F.

1 *Hypotheſis.* Let A ⊏ C, becauſe [a] E. F :: B. C, by [b] inverſion ſhall be F. E :: C. B. [c] But $\frac{C}{B} \sqsupset \frac{A}{B}$, [d] therefore $\frac{F}{E} \sqsupset \frac{A}{B}$ or $\frac{D}{E}$, [e] therefore D ⊏ F. Which was to be demonſtrated.

[a] *hyp.*
[b] *cor.* 4. 5.
[c] *hyp.* & 8. 5.
[d] *ſch.* 13. 5
[e] 10. 5.

2 *Hypotheſis.* By the ſame way of argument if A ⊐ C, it will appear that D ⊐ F.

3 *Hypotheſis.* If A = C, becauſe F. E :: C. B :: [f] A. B :: D. E, [g] therefore is D = F. Which was to be demonſtrated.

[f] 7. 5.
[g] 11. 5. & 9. 5.

PROP. XXI.

If there be three magnitudes A, B, C, and others alſo D, E, F, equal to them in number; which taken two and two are in the ſame ratio, and their proportion inordinate (A. B :: E. F, and B. C :: D. E) and if of equality the firſt A be greater than the third C, then is the fourth D greater than the ſixth F: but if the firſt be equal to the third, then is the fourth equal to the ſixth; if leſs, ſo is the other likewiſe.

1 *Hypotheſis.* If A ⊏ C, then becauſe [a] D. E :: B. C, therefore inverſly E. D :: C. B. But $\frac{C}{B} \sqsupset \frac{A}{B}$, [b] [c] therefore $\frac{E}{D} \sqsupset \frac{A}{B}$: that is, $\frac{E}{F}$. [d] Therefore D ⊏ F.

[a] *hyp.*
[b] 8. 5.
[c] *ſch.* 13. 5.
[d] 10. 5.

2 *Hypotheſis.* By the like argument if A ⊐ C, then is D ⊐ F.

3 *Hypotheſis.* If A = C, then becauſe E. D :: [e] C. B :: [e] A. B :: [f] E. F, [g] therefore is D = F. Which was to be demonſtrated.

[e] 7. 5.
[f] *hyp.*
[g] 9. 5.

PROP. XXII.

If there be any number of magnitudes A,B,C, and others equal to them in number D,E,F, which taken two and two are in the same ratio (A.B :: D.E, and B.C :: E.F) they shall be in the same ratio also by equality (A.C :: D.F.)

Take G,H equimultiplices of A,D, and I,K of B,E, and also L,M of E,F.

[a] *hyp.* [b] 4.5. [c] 20.5. [d] 6 *def.* 5.

Because [a] A.B :: D.E, [b] therefore G.I :: H.K, and in like manner I.L :: K.M; therefore if G⊏, =, ⊐L, [c] then is H⊏, =, ⊐M, [d] therefore A.C :: D.F. By the same way of demonstration if further C.N :: F.O, then by equality A.N :: D.O. Which was to be demonstrated.

PROP. XXIII.

If there be three magnitudes A,B,C, and others D,E,F, equal to them in number, which taken two and two art in the same ratio, and their proportionality inordinate (A.B :: E.F, and B.C :: D,E) they shall be in the same ratio also by equality.

Take G,H,I equimultiplices of A,B,D, and also K,L,M equimultiplices of C,E,F; then G.H :: [a] A.B :: [b] E.F [a] :: L.M. Moreover, because [b] B.C :: D.E, thence is [c] H.K :: I.L; therefore G, H, K and I, L, M are according to 21.5. Wherefore if G be ⊏, =, ⊐ K, then is likewise I ⊏, =, ⊐ M; and so [d] consequently A.C :: D.F. Which was to be demonstrated.

[a] 15.5. [b] *hyp.* [c] 4.5. [d] 6 *def.* 5.

If there be more magnitudes than three, this way of demonstration holds good in them also.

Co-

Corollarium.

From hence * it follows, that ratio's compounded of the ſame ratio's, are among themſelves the ſame; as alſo that the ſame parts of the ſame ratio's are among themſelves the ſame. * 22 & 23 5 & 20 *def.* 5.

PROP. XXIV.

If the firſt magnitude AB have the ſame ratio to the ſecond C, which the third DE hath to the fourth F; and if the fifth BG have the ſame ratio to the ſecond C, which the ſixth EH hath to the fourth F: then ſhall the firſt compounded with the fifth (AG) have the ſame ratio to the ſecond C, which the third compounded with the ſixth (DH) hath to the fourth F.

For becauſe [a] AB.C :: DE.F, and by the Hypotheſis and inverſion C.BG :: F.EH; therefore by [b] equality AB.BG :: DE.EH, whence by compounding AG.BG :: DH.EH, alſo [c] BG.C :: EH.F. Therefore again by [b] equality AG.C :: DH.F. Which was to be demonſtrated.

[a] *hyp.* [b] 22.5. [c] *hyp.*

PROP. XXV.

If four magnitudes be proportional (AB.CD :: E.F) the greateſt AB and the leaſt F ſhall be greater than the reſt CD and E.

Make AG = E and CH = F, becauſe AB.CD :: [a] E.F :: [b] AG.CH, [c] thence is AB.CD :: GB.HD. [d] But AB ▭ CD, [e] therefore GB ▭ HD; but AG + F = E + CH, therefore AG + F + GB ▭ E + CH + HD, that is, AB + F ▭ E + CD. Which was to be demonſtrated.

[a] *hyp.* [b] 7.5. [c] 19.5. [d] *hyp.* [e] *ſch.* 14.

These Propositions which follow are not Euclid's, *but taken out of other Authors, and here subjoyned because of their frequent use.*

PROP. XXVI.

If the first have a greater proportion to the second, than the third to the fourth; then contrariwise by conversion the second shall have a less proportion to the first, than the fourth to the third.

Let $\frac{A}{B} \sqsubset \frac{C}{D}$, I say that $\frac{B}{A} \sqsupset \frac{D}{C}$, for conceive $\frac{C}{D} = \frac{E}{B}$, [a] therefore $\frac{A}{B} \sqsubset \frac{E}{B}$, [b] whence $A \sqsubset E$. [c] Therefore $\frac{B}{A} \sqsupset \frac{B}{E}$ [d] or $\frac{D}{C}$. Which was to be demonstrated.

[a] 13.5. [b] 10.5. [c] 8.5. [d] *cor.* 4.5.

PROP. XXVII.

If the first have a greater proportion to the second, than the third to the fourth; then discretely the first shall have a greater proportion to the third, than the second to the fourth.

Let $\frac{A}{B} \sqsubset \frac{C}{D}$, then I say $\frac{A}{C} \sqsubset \frac{B}{D}$, for conceive $\frac{E}{B} = \frac{C}{D}$, [a] therefore $A \sqsubset E$, [b] and therefore $\frac{A}{C} \sqsubset \frac{E}{C}$ [c] or $\frac{B}{D}$. Which was to be demonstrated.

[a] 10.5. [b] 8.5. [c] 16.5.

PROP. XXVIII.

If the first have a greater proportion to the second, than the third to the fourth; then the first compound-

ed

ed with the second, shall have a greater proportion to the second, than the third compounded with the fourth to the fourth.

Let $\frac{AB}{BC} \sqsupset \frac{DE}{EF}$, I say that $\frac{AC}{BC} \sqsupset \frac{DE}{EF}$, for conceive $\frac{GB}{BC} = \frac{DE}{EF}$, [a] therefore is AB ⊐ GB. Add BC to each part, then [b] will AC ⊐ GC, [c] therefore $\frac{AC}{EC} \sqsupset \frac{GC}{BC}$, [d] that is, $\frac{DF}{FE}$. Which was to be demonstrated.

[a] 10.5. [b] 4 *ax.* [c] 8.5. [d] 18.5.

PROP. XXIX.

If the first compounded with the second have a greater proportion to the second, than the third compounded with the fourth hath to the fourth; then by division the first shall have a greater proportion to the second, than the third to the fourth.

Let $\frac{AC}{BC} \sqsupset \frac{DF}{EF}$, then I say $\frac{AB}{EC} \sqsupset \frac{DE}{EF}$, for conceive $\frac{GC}{EC} = \frac{DE}{EF}$, [a] therefore AC ⊐ GC. Take away EC that is common, then [b] remains AB ⊐ GB, [c] therefore $\frac{AB}{BC} \sqsupset \frac{GB}{EC}$ [d] or $\frac{DE}{EF}$. Which was to be demonstrated.

[a] 10.5. [b] 5 *ax.* [c] 8.5. [d] 17.5.

PROP. XXX.

If the first compounded with the second have a greater proportion to the second, than the third compounded with the fourth hath to the fourth; then by converse ratio shall the first compounded with the second have a lesser ratio to the first, than the third compounded with the fourth shall have to the third.

These Propositions which follow are not Euclid's, *but taken out of other Authors, and here subjoyned because of their frequent use.*

PROP. XXVI.

If the first have a greater proportion to the second, than the third to the fourth; then contrariwise by conversion the second shall have a less proportion to the first, than the fourth to the third.

Let $\frac{A}{B} \sqsubset \frac{C}{D}$, I say that $\frac{B}{A} \sqsupset \frac{D}{C}$, for conceive $\frac{C}{D} = \frac{E}{B}$, [a] therefore $\frac{A}{B} \sqsubset \frac{E}{B}$, [b] whence $A \sqsubset E$. [c] Therefore $\frac{B}{A} \sqsupset \frac{B}{E}$ [d] or $\frac{D}{C}$. Which was to be demonstrated.

[a] 13.5. [b] 10.5. [c] 8.5. [d] *cor.* 4.5.

PROP. XXVII.

If the first have a greater proportion to the second, than the third to the fourth; then alternately the first shall have a greater proportion to the third, than the second to the fourth.

Let $\frac{A}{B} \sqsubset \frac{C}{D}$, then I say $\frac{A}{C} \sqsubset \frac{B}{D}$, for conceive $\frac{E}{B} = \frac{C}{D}$, [a] therefore $A \sqsubset E$, [b] and therefore $\frac{A}{C} \sqsubset \frac{E}{C}$ [c] or $\frac{B}{D}$. Which was to be demonstrated.

[a] 10.5. [b] 8.5. [c] 16.5.

PROP. XXVIII.

If the first have a greater proportion to the second, than the third to the fourth; then the first compounded

ed with the ſecond, ſhall have a greater proportion to the ſecond, than the third compounded with the fourth to the fourth.

Let $\frac{AB}{BC} \sqsupset \frac{DE}{EF}$, I ſay that $\frac{AC}{BC} \sqsupset \frac{DE}{EF}$, for conceive $\frac{GB}{BC} = \frac{DE}{EF}$, [a] therefore is AB ⊐ GB. Add BC to each part, then [b] will AC ⊐ GC, [c] therefore $\frac{AC}{EC} \sqsupset \frac{GC}{EC}$, [d] that is, $\frac{DF}{FE}$. Which was to be demonſtrated.

[a] 10. 5. [b] 4 *ax.* [c] 8. 5. [d] 18. 5.

PROP. XXIX.

If the firſt compounded with the ſecond have a greater proportion to the ſecond, than the third compounded with the fourth hath to the fourth; then by diviſion the firſt ſhall have a greater proportion to the ſecond, than the third to the fourth.

Let $\frac{AC}{BC} \sqsupset \frac{DF}{EF}$, then I ſay $\frac{AB}{EC} \sqsupset \frac{DE}{EF}$, for conceive $\frac{GC}{EC} = \frac{DE}{EF}$, [a] therefore AC ⊐ GC. Take away EC that is common, then [b] remains AB ⊐ GB, [c] therefore $\frac{AB}{BC} \sqsupset \frac{GB}{EC}$ [d] or $\frac{DE}{EF}$. Which was to be demonſtrated.

[a] 10. 5. [b] 5 *ax.* [c] 8. 5. [d] 17. 5.

PROP. XXX.

If the firſt compounded with the ſecond have a greater proportion to the ſecond, than the third compounded with the fourth hath to the fourth; then by converſe ratio ſhall the firſt compounded with the ſecond have a leſſer ratio to the firſt, than the third compounded with the fourth ſhall have to the third.

Let $\frac{AC}{BC} \sqsubset \frac{DF}{EF}$, then I say that $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$.

[a] *hyp.* For becauſe that $\frac{AC}{BC}$ [a] $\sqsubset \frac{DF}{EF}$. [b] Therefore by

[b] 29.5. diviſion $\frac{AB}{BC} \sqsubset \frac{DE}{EF}$, by converſion [c] therefore

[c] 26.5. $\frac{BC}{AB} \sqsupset \frac{EF}{DE}$, and [d] therefore by compoſition

[d] 28.5. $\frac{AC}{AB} \sqsupset \frac{DF}{DE}$. Which was to be demonſtrated.

PROP. XXXI.

If there be three magnitudes A,B,C, and others alſo D,E,F equal to them in number; and if there be a greater proportion of the firſt of the former to the ſecond, than there is of the firſt of the laſt to their ſecond $\left(\frac{A}{B} \sqsubset \frac{D}{E}\right)$; *and there be alſo a greater proportion of the ſecond of the firſt magnitudes to the third, than there is of the ſecond of the laſt magnitudes to their third* $\left(\frac{B}{C} \sqsubset \frac{E}{F}\right)$. *Then by equality alſo ſhall the ratio of the firſt of the former magnitudes to the third be greater, than the ratio of the firſt of the latter magnitudes to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{F}\right)$.

[a] 10.5. Conceive $\frac{G}{C} = \frac{E}{F}$, [a] therefore is B $\sqsubset$ G, and

[b] 8.5. [b] therefore $\frac{A}{G} \sqsubset \frac{A}{B}$. Again conceive $\frac{H}{G} = \frac{D}{E}$

[c] 13.5. [c] therefore $\frac{H}{G} \sqsupset \frac{A}{B}$; therefore much more

$\frac{H}{G} \sqsupset \frac{A}{G}$. [d] Wherefore $A \sqsubset H$, [e] and consequently $\frac{A}{C} \sqsubset \frac{H}{C}$. [f] But $\frac{H}{C} = \frac{H}{G} + \frac{G}{C} = [g] \frac{D}{E} + \frac{E}{F} = \frac{D}{F}$. [h] Therefore $\frac{A}{C} \sqsubset \frac{D}{F}$.

[d] 10. 5.
[e] 8. 5.
[f] *def.* 20. 5.
[g] *supra.*
[h] *sch.* 13. 5.

PROP. XXXII.

If there be three magnitudes A, B, C, and others D, E, F, equal to them in number; and there be a greater proportion of the first of the former magnitudes to the second, than there is of the second of the latter to the third $\left(\frac{A}{B} \sqsubset \frac{E}{F}\right)$ *and also the ratio of the second of the former to the third, be greater than the ratio of the first of the latter to the second* $\left(\frac{B}{C} \sqsubset \frac{D}{E}\right)$ *then by equality also shall the proportion of the first of the former to the third, be greater than that of the first of the latter to the third* $\left(\frac{A}{C} \sqsubset \frac{D}{F}\right)$.

Suppose $\frac{G}{C} = \frac{D}{E}$, therefore is [a] $B \sqsubset G$, and therefore [b] $\frac{A}{G} \sqsubset \frac{A}{B}$. Again suppose $\frac{H}{G} = \frac{E}{F}$ therefore is [c] $\frac{H}{G} \sqsupset \frac{A}{G}$, and consequently [a] $A \sqsubset H$, and thence [b] $\frac{A}{C} \sqsubset \frac{H}{\quad}$ or $\frac{D}{F}$. Which was to be demonstrated.

[a] 10. 5.
[b] 8. 5.
[c] *sch.* 13. 5.
[d] 23. 5.

PROP.

PROP. XXXIII.

If the proportion of the whole AB to the whole CD, be greater than the proportion of the part taken away AE, to the part taken away CF; then shall also the ratio of the remainder EB to the remainder FD, be greater than that of the whole AB to the whole CD.

Because that $\frac{AB}{CD}$ [a] $\sqsupset$ $\frac{AE}{CF}$, [b] therefore by permutation $\frac{AB}{AE}$ $\sqsupset$ $\frac{CD}{CF}$, [c] therefore by converse ratio $\frac{AB}{EB}$ $\sqsupset$ $\frac{CD}{FD}$, and by permutation again $\frac{AB}{CD}$ $\sqsupset$ $\frac{EB}{FD}$. Which was to be demonstrated.

[a] *by p.*
[b] 27. 5.
[c] 30. 5.

PROP. XXXIV.

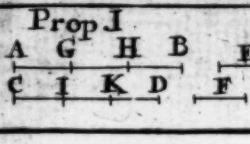
If there be any number of magnitudes, and others also equal to them in number; and the proportion of the first of the former to the first of the latter, be greater than that of the second to the second, and that greater than the proportion of the third to the third, and so forward: all the former magnitudes together shall have a greater ratio to all the latter together, than all the former, leaving out the first, shall have to all the latter, leaving out the first; but less than that of the first of the former to the first of the latter: and lastly, greater than that of the last of the former to the last of the latter.

You may please to consult Interpreters for the demonstration hereof, we having for brevities sake omitted it, and because 'tis of no use in these Elements.

The End of the Fifth Book.

The Fifth Book of Euclid

Prop I



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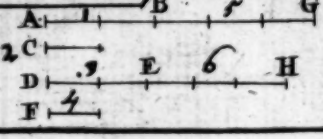

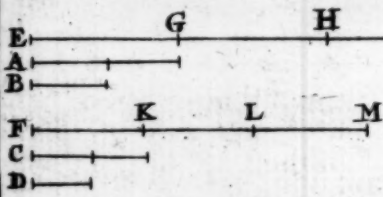

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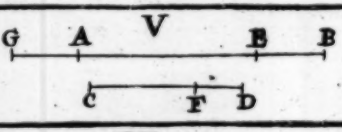

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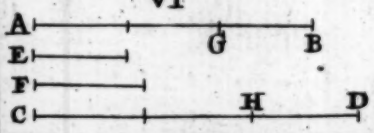

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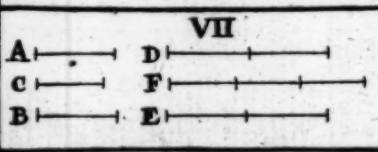

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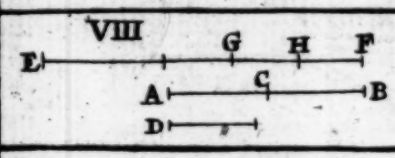

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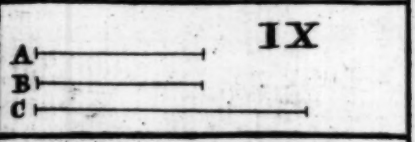

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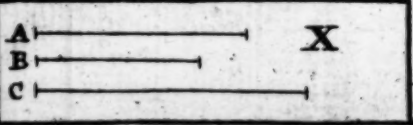
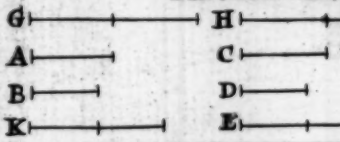


XI. XII. XI



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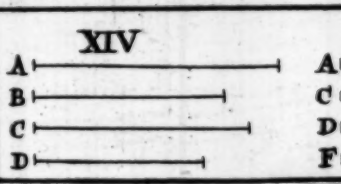

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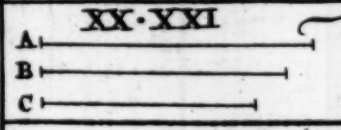
XVIII

XX. XXI



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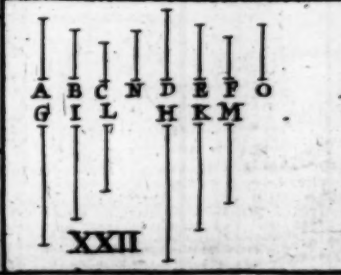

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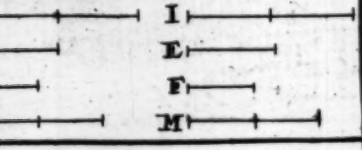

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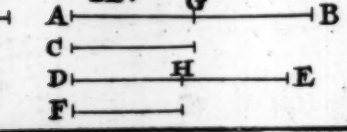
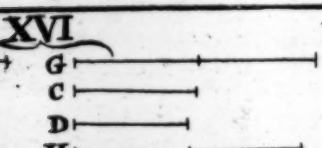
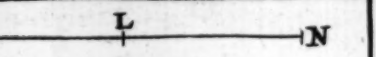


XVI





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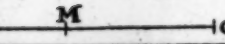

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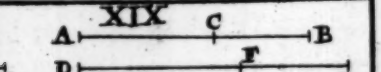

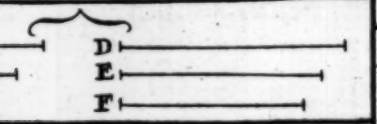

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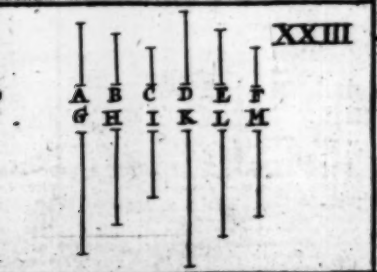

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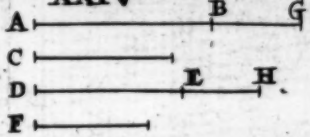
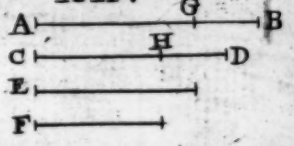
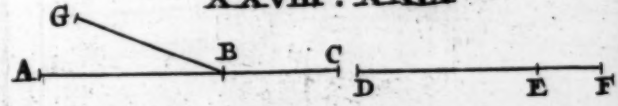


XXV



XXVI. XXVII

XXVIII. XXIX



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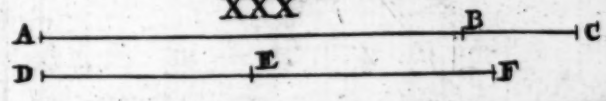
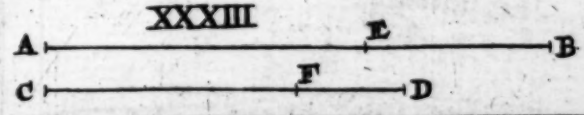


XXXI. XXXII

XXXIII



XXXIV

Place this at the end of the Fifth Book.

Tho Cross Junior Sculp.

THE SIXTH BOOK OF Euclid's *Elements*.

Definitions.

I. The right lined figures (ABC, DCE) are ſuch whoſe ſeveral angles are equal one to the other, and alſo their ſides about the equal angles proportional.

The angle B = DCE, and AB. BC :: DC. CE. Alſo the angle A = D, and BA. AC :: CD. DE. Laſtly, the angle ACB = E, and BC. CA :: CE. ED.

II. Reciprocal figures are (BD, BF) when in both the figures are the terms antecedents and conſequents of ratio's (that is, AB. BG :: EB. BC.)

III. A right line AB is ſaid to be cut according to mean and extreme proportion, when as the whole AB is to the greater ſegment AC, ſo is the greater ſegment AC to the leſs CB (AB. AC :: AC. CB.)

IV.

IV. The altitude of any figure ABC is a perpendicular line AD, drawn from the top A to the base BC.

V. A ratio is said to be compounded of two ratio's, when the quantities of the ratio's being multiplied the one into the other, do produce any ratio.

As the ratio of A to C is compounded of the ratio's of A to B and B to C: for,

$$\frac{A}{B} + \frac{B}{C} \overset{a}{=} \frac{A}{C} \overset{b}{=} \frac{AB}{BC}.$$

a 20 *def.* 5.
b 15. 5.

PROP. I.

Triangles ABC, ACD, and parallelograms BCAE, CDFA, which have the same height, are in proportion one to the other as their bases BC, CD are.

a 3. 1. a Take as many as you please BG, GH equal to BC, and also DI = CD, and joyn AG, AH, AI.

b 38. 1. b The triangles ACB, ABG, AGH are equal, and b also the triangle ACD = ADI; therefore the triangle ACH is as multiplex of the triangle ACB, as the base HC is of the base BC; and the triangle ACI as multiplex of the triangle ACD, as the base CI is of CD. But if HC $\sqsubset$, $=$, $\sqsupset$ CI, c then is likewise the triangle AHC $\sqsubset$, $=$, $\sqsupset$ ACI, and d therefore BC. CD :: the triangle ABC. ACD :: e parallelogram CE. CF. Which was to be demonstrated.

c *sch.* 38. 1.
d 6 *def.* 5.
e 41. 1. & 15. 5.

Scholium.

Hence triangles ABC, DEF, and parallelograms AGBC, DEFH, whose bases BC, EF are equal, are in such proportion as their altitudes AI, DK are.

a Ta ke

[a] Take IL=CB, and KM=EF, and joyn LA,LG,MD,MH; then it is evident that the triangle ABC.DEF :: [b] ALI. DKM :: [c] AI. DK :: [d] parallelograms AGBC,DEFH. Which was to be demonſtrated.

[a] 3.1.
[b] 7.5.
[c] 1.6.
[d] 41.1.& 15.5.

PROP. II.

If to one ſide BC of a triangle ABC be drawn a parallel right line DE, the ſame ſhall cut the ſides of the triangle proportionally (AD. BD :: AE. EC.) And if the ſides of the triangle be proportionally cut (AD . BD :: AE.EC) then a right line DE joyned at the ſections D, E, ſhall be parallel to the remaining ſide of the triangle BC: draw CD and BE.

1 *Hypotheſis.* Becauſe the triangle DEB [a] = DEC, [b] therefore ſhall be the triangle ADE. DBE :: ADE. ECD. But the triangle ADE. DBE :: [c] AD. DB, and the triangle ADE. DEC :: AE.EC, [d] therefore AD.DB :: AE.EC.

[a] 37. 1.
[b] 7.5.
[c] 1.6.
[d] 11.5.

2 *Hypotheſis.* Becauſe AD. DB :: AE. EC, [e] that is the triangle ADE.DBE :: ADE.ECD; [f] therefore is the triangle DBE = ECD, and [g] therefore DE,BC are parallels. Which was to be demonſtrated.

[e] 1.6.
[f] 9.5.
[g] 39.1.

Scholium.

If there be drawn many parallels DE,FG to one ſide BC of any triangle, then all the ſegments of the ſides ſhall be proportional; for DF.FA :: EG.GA, then by compounding and inverting 'tis FA.DA :: GA.EA, [h] and DA.DB :: EA. EC. Therefore by equality DF. DB :: EG.EC. Which was to be demonſtrated.

[h] 2.6.

Co-

Corollarium.

If DF.DB :: EG. EC, then are [h] BC, DE, FG parallel.

PROP. III.

If an angle BAC of a triangle BAC be biſected, and the right line AD, that biſects the angle, cut the baſe alſo; then ſhall the ſegments of the baſe have the ſame ratio that the other ſides of the triangle have, (BD.DC :: AB.AC.) And if the ſegments of the baſe have the ſame ratio, that the other ſides of the triangle have (BD.DC :: AB. AC) then a right line AD, drawn from the top A to the ſection D, ſhall biſect that angle BAC of the triangle.

Produce BA, and make AE = AC, and joyn CE.

1 *Hypotheſis.* Becauſe AE = AC, therefore
[a] 5.1. is the angle ACE [a] = E [b] = ½BAC [c] = DAC,
[b] 32.1. [d] therefore DA, CE are parallels. [e] Wherefore
[c] *hyp.* BA. AE(AC) :: BD.DC.
[d] 27.1. 2 *Hypotheſis.* Becauſe BA. AC(AE) :: BD.
[e] 2.6. DC, [f] therefore are DA, CE parallels, and
[f] 2.6. [g] therefore is the angle BAD = E, and the an-
[g] 29.1. gle DAC [g] = ACE [h] = E, [k] therefore the an-
[h] 5.1. gle BAD = DAC. Wherefore the angle BAC
[k] 1 *ax.* is biſected. Which was to be demonſtrated.

PROP. IV.

Of equiangular triangles ABC, DCE, the ſides are proportional which are about the equal angles B, DCE, (AB.BC :: DC.CE, &c.) and the ſides AB, DC, &c. which are ſubtended under the equal angles ACB, E, &c. are homologous or of like ratio.

Set

Set the side BC in a direct line to the side CE, and produce BA and ED till they [a] meet. [a] 32.1.&

Because the angle B [b] = ECD, [c] therefore BF, CD are parallel. Also because the angle BCA [b] = CED, [c] therefore are CA, EF parallel. Therefore the figure CAFD is a parallelogram, [d] therefore AF = CD, and AC = [d] FD. Whence it is evident that AB. AF (CD) :: [e] BC. CE, [f] by permutation therefore AB. BC :: CD. CE; also BC. CE :: FD (AC). DE, [f] and thence by permutation BC. AC :: CE. DE. [g] Wherefore also by equality AB. AC :: [g] CD. DE. Therefore, &c. [b] *hyp.* [c] 28.1. [d] 34.1. [e] 2.6. [f] 16.5. [g] 22.5.

Corollarium.

Hence AB. DC :: BC. CE :: AC. DE.

Scholium.

Hence if in a triangle FBE there be drawn AC a parallel to one side FE, the triangle ABC shall be like to the whole FBE.

PROP. V.

If two triangles ABC, DEF have their sides proportional (AB. BC :: DE. EF, and AC. BC :: DF. EF, and also AB. AC :: DE. DF) those triangles are equiangular, and those angles equal under which are subtended the homologous sides.

At the side EF [a] make the angle FEG = B, and the angle EFG = C, [b] whence the angle G = A. Therefore GE. EF [c] :: AB. BC :: [d] DE. EF, [c] and therefore GE = DE; likewise GF. FE [c] :: AC. CB :: [d] DF. FE, [e] therefore GF = DF. Therefore the triangles DEF, GEF are mu- [a] 23.1. [b] 32.1. [c] 4.6. [d] *hyp.* [e] 11.5 9.5.

mutually equilateral, [f] therefore the angle D =G=A, and the angle FED [f] = FEG=B, and [g] consequently the angle DFE=C. Therefore, &c.

[f] 8.1.
[g] 32.1.

PROP. VI.

If two triangles ABC, DEF have one angle B equal to one angle DEF, and the sides about the equal angles B, DEF proportional (AB. BC :: DE. EF) then those triangles ABC, DEF are equiangular, and have those angles equal, under which are subtended the homologous sides.

At the side EF make the angle FEG=B, and the angle EFG=C, [a] then will the angle G=A. Therefore GE. EF :: [b] AB. BC :: [c] DE. EF, [d] and therefore DE=GE. But the angle DEF [e] = B [f] = GEF; therefore the angle D [g] = G = A, [h] and consequently the angle EFD =C. Which was to be demonstrated.

[a] 32.1.
[b] 4.6.
[c] *hyp.*
[d] 9.5.
[e] *hyp.*
[f] *constr.*
[g] 4.1.
[h] 32.1.

PROP. VII.

If two triangles ABC, DEF have one angle A equal to one angle D, and the sides about the other angles ABC, E proportional (AB. BC :: DE. EF) and if they have both of the remaining angles C, F either less or not less than a right angle; then shall the triangles ABC, DEF be equiangular, and have those angles equal about which the proportional sides are.

[a] *hyp.*
[b] 32.1.
[c] 4.6.
[d] *hyp.*
[e] 9.5.
[f] 5.1.
[g] *cor.* 17.1
[h] *cor.* 13.1.

For, if it can be, let the angle ABC [a] E, and make the angle ABG=E. Therefore whereas the angle A [a] = D, [b] thence is the angle AGB =F. Therefore AB. BG [c] :: DE. EF :: [d] AB. BC, [e] therefore BG=EC, [f] therefore the angle BGC=BCG. [g] Therefore BGC or C is less than a right angle, and [h] consequently AGB or

F is greater than a right. Therefore the angles C and F are not of the ſame ſpecies or kind. Which is againſt the Hypotheſis.

PROP. VIII.

If in a right angled triangle ABC, from the right angle BAC there be drawn AD a perpendicular to the baſe BC, then the triangles about the perpendicular (ADB, ADC) are like both to the whole triangle ABC, and alſo one to the other.

For becauſe BAC, ADB are [a] right angles, [b] and ſo equal, and B common, the triangles BAC, ADB [c] are like. By the ſame argument BAC, ADC are like, [d] whence alſo ADB, ADC will be like. Which was to be demonſtrated.

[a] *hyp.*
[b] 12 *ax.*
[c] 32. & 4.6.
[d] 21.6.

Corollaria.

Hence 1. BD.DA [e] :: DA.DC. [e] 1 *def.*6.
2. BC.AC :: AC.DC, and CB.BA :: BA.BD.

PROP. IX.

From a right line given AB to cut off any part required, as ⅓ (AG.)

From the point A draw an infinite line AC any wiſe, in which [a] take any three equal parts AD, DE, EF, joyn FB, to which from D [b] draw the parallel DG, and the thing is done.

For GB.AG :: [c] FD.AD, whence by [d] compoſition AB.AG :: AF.AD. Therefore whereas AD = ⅓ of AF, therefore is AG = ⅓ of AB. Which was to be done.

[a] 3.1.
[b] 31.1.
[c] 2.6.
[d] 18.5.

PROP.

PROP. X.

To divide a right line given AB not divided (in F and G) as another line given AC was cut (in D and E.)

Let a right line BC joyn the extremities of the line divided, and of the line not divided; and to that line from the points E, D [a] draw the parallels EG, DF meeting with the right line that is to be cut in G and F, then the thing is done.

[a] 31.1.

For let DH be [a] drawn parallel to AB, then AB. DE :: [b] AF. FG, and DE. EC [b] :: DI. IH :: [c] FG. GB. Which was to be done.

[b] 2.6. [c] 34.1. & 7.5.

Scholium.

Hence is learnt to cut a right line given AB into as many equal parts as you please (suppose 5) which will be more easily performed thus.

Draw an infinite line AD, and another BH parallel to it, and infinite also. Of those take equal parts AR, RS, SV, VN, and BZ, ZX, XT, TL, in each line less parts by one than are required in AB; then let the right lines LR, TS, XV, ZN be drawn, these lines so drawn shall cut the right line given AB into five parts.

For RL, ST, VX, NZ are [a] parallels, therefore whereas AR, RS, SV, VN are [b] equal, [c] thence AM, MO, OP, PQ are equal also. Likewise because that BZ = ZX, therefore is BQ = PQ, and therefore AB is cut into five parts. Which was to be done.

[a] 33.1. [b] *constr.* [c] 2.6.

PROP.

PROP. XI.

Two right lines being given AB, AD, to find out a third in proportion to them (DE.)

Joyn BD, and from AB being produced take BC=AD; through C draw CE parallel to BD, with which let AD produced meet in E, then is DE the proportional required.

For AB.BC (AD) [a] :: AD.DE. Which was to be done. [a] 2.6.

Or thus: Let the given lines be BD, AD, with which make the right Angle BDA. Draw BA, and to this the Perpendicular AC, with which let BD produced meet in C, and 'tis [b] BD.DA :: DA.DC. *See Fig. Prop.* 8. V [b] *Cor.* 8.6.

PROP. XII.

Three right lines being given DE, EF, DG, to find out a fourth proportional GH.

Joyn EG, and through F draw FH parallel to EG; with which let DG produced to H meet. Then it is evident that DE.EF [a] :: DG.GH. Which was to be done. [a] 2.6.

PROP. XIII.

Two right lines being given AE, EB, to find out a mean proportional EF.

Upon the whole line AB as a diameter describe a semicircle AFB, and from E erect a perpendicular EF meeting with the periphery in F. then AE.EF :: EF.EB. For let AF and FB be drawn; [a] then from the right angle of the right-angled triangle AFB is drawn a right line FE perpendicular to the base. [a] 31.3.

[b] There-

[b] *cor.* 8. 6. [b] Therefore AE. FE :: FE. EB. Which was to be done. Or (in the same figure) let the two given lines be AB, BE, 'tis evident that AB. BF :: BE. BE.

Corollarium.

Hence, A right line drawn in a circle from any point of the diameter perpendicularly, and extended to the circumference, is a mean proportional betwixt the two segments of the diameter.

PROP. XIV.

Equal Parallelograms having one angle ABC equal to one EBG, have the sides BD, BF which are about the equal angles reciprocal (AB. BG :: EB. BC;) and those Parallelograms BD, BF which have one angle ABC equal to one EBG, and the sides which are about the equal angles reciprocal, are equal.

[a] *sch.* 15. 1. [b] 1. 6. [c] 7. 5. [d] 1. 6. [e] 11. 5. [f] 1. 6. [g] *hyp.* [h] 1. 6. [k] 11, & 9. 5.

For let the sides AB, BG about the equal angles make one right line; [a] wherefore EB, BC shall do the same. Let FG, DC be produced till they meet.

1. *Hypothesis.* AB. BG [b] :: BD. BH :: [c] BF. BH :: [d] BE. EC. [e] therefore, &c.

2. *Hypothesis.* BD. BH :: [f] AB. BG :: [g] BE. BC :: [h] BF. BH. [k] Therefore the Parallelogram BD = BF. Which was to be demonstrated.

PROP. XV.

Equal triangles having one angle ABC equal to one DBE, their sides which are about the equal angles are reciprocal (AB. BE :: DB. BC.) And those

thoſe triangles that have one angle ABC equal to one DBE, and have alſo the ſides that are about the equal angles reciprocal (AB. BE :: DB. BC.) are equal.

Let the ſides CB, BD, which are about the equal angles be ſet in a ſtrait line; [a] therefore ABE is a right line. Let CE be [b] drawn.

1. *Hypotheſis.* AB. BE :: [b] the triangle ABC. CBE [c] :: the triangle DBE. CBE :: [d] DB. BC. [e] therefore, &c.

2. *Hypotheſis.* The triangle ABC. CBE :: [f] AB. BE :: [g] DB. BC [h] :: the triangle DBE. CBE. [i] Therefore the triangle ABC = DBE. Which was to be demonſtrated.

[a] *ſch.* 15. 1.
[b] 1. 6.
[c] 7. 5.
[d] 1. 6.
[e] 11. 5.
[f] 1. 6.
[g] *hyp.*
[h] 1. 6.
[k] 11, & 9. 5.

PROP. XVI.

If four right lines be proportional (AB. FG :: EF. CB.) the rectangle AC comprehended under the extremes AB, CB, is equal to the rectangle EG comprehended under the means FG, EF. And if the rectangle AC comprehended under the extremes AB, CB be equal to the rectangle EG comprehended under the means FG, EF, then are the four right lines proportional. (AB. FG :: EF. CB.)

1. *Hypotheſis.* The angles B and F are right, and [a] conſequently equal, and by hypotheſis AB. FG :: EF. CB. [b] therefore the rectangle AC = EG.

2. *Hypotheſis.* The rectangle AC [c] = EG, and the angle B = F; [d] therefore AB. FG :: EF. CB. Which was to be demonſtrated.

[a] 12 *ax.*
[b] 14. 6.
[c] *Hyp.*
[d] 14. 6.

Corol.

Corollarium.

Hence, it is easie to apply a rectangle given EG to a right line given AB; (*viz.*) by [c] making AB. EF :: FG. BC.

c 12. 6.

PROP. XVII.

If three right lines be proportional (AB. EF :: EF. CB.) the rectangle AC made under the extreams AB, CB is equal to the square EG made of the middle EF. And if the rectangle AC comprehended under the extreams AB, CB be equal to the square EG made of the middle EF, then the three lines are proportional. (AB. EF :: EF. CB.)

Take FG = EF.

a *hyp.* 1. *Hypothesis.* AB. EF :: [a] EF (FG). CB.
b 16. 6. therefore the rectangle AC [b] = EG [c] = EFq.
c 29 *def.* 1. 2. *Hypothesis.* The rectangle AC [d] = to the
d *hyp.* square EG = EFq. [e] therefore AB. EF :: FG
e 16. 6. (EF). BC. Which was to be demonstrated.

Corollarium.

Let A × B = Cq. therefore A. C :: C. B.

PROP. XVIII.

From a right line given AB to describe a right-lined figure AGHB like and alike situate to a right-lined figure given CEFD.

Resolve the right-lined figure given into
a 23. 1. triangles; [a] Make the angle ABH = D, [a] and the angle BAH = DCF, [a] and the angle AHG = CFE, [a] and the angle HAG = FCE, then AGHB shall be the right-lined figure sought. For

For the angle B [b]=D, and the angle BAH [b] =DCF. [c] wherefore the angle AHB=CFD. [b] alſo the angle HAG = FCE, and the angle AHG [b] =CFE. [c] wherefore the angle G=E, and the whole angle GAB [d] = ECD, and the whole angle GHB [d] = EFD. The polygones therefore are mutually equiangular. Moreover becauſe the triangles are equiangular, therefore AB . BH [e] :: CD . DF and AG . GH [e] :: CE . EF. Likewiſe AG . AH :: ECF . CF. and AH . AB :: CF . CD [f] From whence by equality AG . AB :: CE . CD After the ſame manner GH . HB :: EF . FD. Therefore the Polygones ABHG, CDFE are like and alike ſituate. Which was to be done.

[b] *conſtr.*
[c] 32. 1.
[d] 2 *ax.*
[e] 4. 6.
[f] 22. 5.
[g] 6 *def.* 5.

PROP. XIX.

Like triangles ABC, DEF are in duplicate ratio of their homologous ſides, BC, EF.

[a] Let there be made BC . EF :: EF . BG, and let AG be drawn. Becauſe that AB . DE [b] :: BC . EF [c] :: EF . BG, and the angle B = E. [d] therefore is the triangle ABG = DEF. But the triangle ABC . ABG :: [e] BC . BG, and $\frac{BC}{BG} = \frac{BC}{EF}$ twice; therefore $\frac{ABC}{ABG}$ that is $\frac{ABC}{DEF}$ [g] $= \frac{BC}{EF}$ twice. Which was to be demonſtrated.

[a] 11. 6.
[b] *cor.* 4. 6.
[c] *conſtr.*
[d] 15. 6.
[e] 1. 6.
[f] 10 *def.* 5
[g] 11. 5.

Corollarium.

Hence, If three right lines (BC, EF, BG) be proportional, then as the firſt is to the third, ſo is a triangle made upon the firſt BC, to a triangle like and alike deſcribed upon the ſecond EF: or ſo is a triangle deſcribed upon

the ſecond EF to a triangle like and alike deſcribed upon the third.

PROP. XX.

Like Polygones ABCDE, FGHIK are divided into equal triangles ABC, FGH, and ACD, FHI, and ADE, FIK; both equal in number and homologous to the wholes (ABC . FGH :: ABCDE . FGHIK :: ACD . FHI :: ADE . FIK.) And the Polygones ABCDE, FGHIK have a double ratio one to the other of what one homologous ſide BC hath to the other homologous ſide GH.

1. For the angle B [a] = G, and AB . BC [a] :: FG . GH, [b] therefore the triangles ABC, FGH are equiangular. After the ſame manner are the triangles AED, FKI like. Whereas therefore the angle BCA [b] = GHF, and the angle ADE [b] = FIK, and the whole angles BCD, GHI, and alſo the whole angles CDE, HIK are [c] equal, there remains the angle ACD [d] = FHI, and the angle ADC = FIH; [e] from whence alſo the angle CAD = HFI. therefore the triangles ACD, FHI are like. Therefore, &c.

[a] *hyp.*
[b] 6. 6.
[b] 6. 6.
[c] *hyp.*
[d] 3 *ax.*
[e] 32. 1.

2. Becauſe that the triangles BCA, GHF are like, [f] therefore is $\frac{BCA}{GHF} = \frac{BC}{GH}$ twice. For the ſame reaſon is $\frac{CAD}{HFI} = \frac{CD}{HI}$ twice; laſtly, $\frac{DEA}{IKF} = \frac{DE}{IK}$ twice. Now whereas that BC. GH [g] :: CD . HI [g] :: DE . IK. [h] therefore is the triangle BCA . GHF :: CAD . HFI :: DEA. IKF [k] :: the polygone ABCDE . FGHIK :: $\frac{BC}{GH}$ twice.

[f] 19. 6.
[g] *hyp.* & 16. 5.
[h] *ſch.* 23. 5.
[k] 12. 5.

Coro!.

Corollaria.

1. Hence, If there be three right lines proportional, then as the firſt is to the third, ſo is a polygone made upon the firſt to a polygone made on the ſecond like and alike deſcribed; or ſo is a polygone deſcribed upon the ſecond to a polygone made on the third like and alike deſcribed.

By which is found out a method of inlarging or diminiſhing any right-lined figure in a ratio given: As if you would make a pentagone quintuple of that pentagone whereof CD is the ſide, then betwixt AB and 5 AB find out a mean proportional, * upon this raiſe a pentagone like to that given, and it ſhall be quintuple of the pentagone given. * 18. 6.

2. Hence alſo, If the homologous ſides of like figures be known, then will the proportion of the figures be evident, *viz.* by finding out a third proportional.

PROP. XXI.

Right-lined figures ABC, DIE which are like to the ſame right-lined figure HFG, are alſo like one to the other.

For the angle A [a] = H [a] = D; and the angle C [a] = G [a] = E; and the angle B [a] = F [a] = I. Alſo [a] AB. AC :: HF. HG :: DI. DE. and [a] AC. CB :: HG. GF :: DE. EI. And AB. BC :: HF. FG :: DI. IE. Therefore [a] ABC, DIE are like. Which was to be demonſtrated. [a] 1 *def.* 6.

PROP. XXII.

If four right lines be proportional (AB. CD :: EF. GH) the right-lined figures alſo deſcribed upon them being like and in like ſort ſituate, ſhall be proportional (ABI. CDK :: EM. GO.) And if the right-lined figures deſcribed upon the lines, like and alike ſituate, be proportional (ABI. CDK :: EM. GO) then the right lines alſo ſhall be proportional (AB. CD :: EF. GH.)

1. *Hypotheſis.* $\frac{ABI}{CDK}$ [a] $= \frac{AB}{CD}$ twice [a] $= \frac{EF}{GH}$ twice [a] $= \frac{EM}{GO}$ [b] therefore ABI. CDK :: EM. GO.

[a] 19.6.

2. *Hypotheſis.* $\frac{AB}{CD}$ twice [a] $= \frac{ABI}{CDK}$, [b] $= \frac{EM}{GO}$ [c] $= \frac{EF}{GH}$ twice. Therefore AB. CD :: EF. GH. Which was to be demonſtrated.

[b] *hyp.*
[c] 20. 6.
[d] *cor.* 23. 5.

Scholium.

Hence is deduced the manner and reaſon of multiplying ſurd quantities. Ex. g. Let $\sqrt{5}$ be to be multiplyed into $\sqrt{3}$. I ſay that the product will be $\sqrt{15}$. For by the definition of Multiplication it ought to be, as 1. $\sqrt{3}$ :: $\sqrt{5}$ to the product. Therefore by *this* q. 1. q. $\sqrt{3}$ :: q. $\sqrt{5}$. q. of the product. That is 1. 3 :: 5 to the ſquare of the product, therefore the ſquare of the product is 15. Wherefore $\sqrt{15}$ is the product of $\sqrt{3}$ into $\sqrt{5}$. Which was to be demonſtrated.

THEOREM.

THEOREM.

If a right line AB be cut any-wiſe in D, the rectangle comprehended under the parts AD, DB is a mean proportional betwixt their ſquares. Likewiſe the rectangle comprehended under the whole AB and one part AD, or DB, is a mean proportional betwixt the ſquare of the whole AB and the ſquare of the ſaid part AD, or DB. Petr. Herig.

Upon the diameter AB deſcribe a ſemicircle; from D erect a perpendicular DE meeting with the periphery in E, joyn AE, BE.

It's evident that AD . DE [a] :: DE . DB [b] therefore ADq . DEq :: DEq . DBq. [c] that is, ADq . ADB :: ADB . DBq. Which was to be demonſtrated.

[a] *Cor.* 8. 6.
[b] 22. 6.
[c] 17. 6.

Moreover BA . AE :: [d] AE . AD. [e] therefore BAq . AEq :: AEq . ADq. [f] that is, BAq . BAD :: BAD . ADq. After the ſame manner ABq . ABD :: ABD . BDq. Which was to be demonſtrated.

[d] *Cor.* 8. 6.
[e] 22. 6.
[f] 17. 6.

Or thus: ſuppoſe Z = A + E. It is manifeſt that Aq . AE :: [a] A . E :: [a] AE . Eq. alſo Zq . ZA :: [a] Z . A :: ZA . Aq. and Zq . ZE :: [a] ZE :: ZE . Eq.

[a] 1. 6.

PROP. XXIII.

Equiangular parallelograms AC, CF, have the ratio one to the other, which is compounded of their ſides.

$$\left(\frac{AC}{CF} = \frac{BC}{CG} + \frac{DC}{CE}\right)$$

[a] *Sch.* 15. 1. Let the sides about the equal angles C be [a] set in a direct line, and let the parallelogram CH be compleated. Then is the ratio of $\frac{AC}{CF}$ [b] $= \frac{AC}{CH} + \frac{CH}{CF}$ [c] $= \frac{BC}{CG} + \frac{DC}{CE}$. Which was to be demonstrated.

[b] 20 *def.* 5.
[c] 1. 6.

Corollaria.

Hence, *and from* 34. 1. *it appears*, 1. *That triangles which have one angle equal (at C) have a ratio compounded of the ratio's of the right lines, (AC to CB, and LC to CF,) containing the equal angle.*

Andr. Tacq. 15. 5.

2. *That all rectangles, and* * *consequently all parallelograms, have their ratio one to the other compounded of the ratio's of base to base, and altitude to altitude.* After the like manner you may argue in triangles.

* 35. 1.

3. *From hence is apparent how to give the proportion of triangles and parallelograms.* Let there be two parallelograms, X, and Z, whose bases are AC, CB, and altitudes CL, CF. Make CL. CF :: CB. O. * then will it be X. Z :: AC. O.

* 14. 6. *and* 1. 6.

PROP. XXIV.

In every parallelogram ABCD, the parallelograms EG, HF which are about the diameter AC are like to the whole, and also one to the other.

For the Parallelograms EG, HF have each of them one angle common with the whole; [a] therefore they are equiangular to the whole, and also one to the other. Also both the triangles ABC, AEI, IHC [a] and the triangles ADC, AGI, IFC are equiangular mutually; [b] there-

[a] 29. 1.

[b] therefore AE. EI :: AB. BC, and [b] AE. AI :: AB. AC, and [b] AI. AG :: AC. AD. [c] Therefore by equality, AE. AG :: AB. AD. [d] Therefore the Parallelograms EG, BD are like. After the same manner are HF, BD like also. Therefore, &c.

[b] 4.6.
[c] 22.5.
[d] 1 *def.* 6.

PROP. XXV.

Unto a right-lined figure given ABEDC to describe another figure P like and alike situate, which also shall be equal to another right-lined figure given F.

[a] Make the rectangle AL = ABEDC; [b] also upon BL make the rectangle BM = F, betwixt AB and BH [c] find out a mean proportional NO: Upon NO [d] make the polygone P like to the right-lined figure given ABEDC. I say, the polygone P so made shall be equal to F that was given.

[a] 45 1.
[b] 44. 1.
[c] 13.6.
[d] 18.6.

For ABEDC (AL). P :: [e] AB. BH :: [f] AL. BM. Therefore P [g] = BM [h] = F. Which was to be done.

[e] *Cor.* 20.6.
[f] 1.6.
[g] 14.5.
[h] *constr.*

PROP. XXVI.

If from a Parallelogram ABCD, be taken away another Parallelogram AGFE, like unto the whole, and in like sort set, having also an angle common with it EAG; then is that parallelogram about the same diagonal AC with the whole.

If you deny AC to be the common diagonal, then let AHC be it, cutting EF in H, and let HI be drawn parallel to AE. Then are Parallelograms EI, DB [a] like. [b] Therefore AE. EH :: AD. DC :: [c] AE. EF. and [d] consequently EH = EF. [f] Which is absurd.

[a] 24.6.
[b] 1 *def.* 6.
[c] *hyp.*
[d] 9.5.
[f] 9 *ax.*

PROP. XXVII.

Of all parallelograms, AD, AG applyed to the ſame right line AB, and wanting in figure by the parallelograms CE, KI like and alike ſet to the parallelogram AD which is deſcribed upon the half, the greateſt is that AD which is applyed to the half being like to the defect KI.

a 43. 1. b 2 ax. c 36. 1. d 2 ax. e 9 ax.

For becauſe that GE [a] = GC, and KI added in common; [b] thence is KE = CI [c] = AM; add CG in common, [d] then is AG = to the Gnomon MBL. But the Gnomon MBL [e] ⊐ CE (AD.) Therefore AG ⊐ AD. Which was to be demonſtrated.

PROP. XXVIII.

Upon a right line given AB, to apply a parallelogram AP equal to a right-lined figure given C, deficient by a parallelogram ZR which is like to another parallelogram given D. * *Now it is requiſite that the right-lined figure given C, whereunto the parallelogram to be applyed must be equal, be not greater than the parallelogram AF which is applyed upon the half line, the defects being like, namely, the defect of the parallelogram AF, which is applyed to the half line, and the defect of the parallelogram AP or D to which it is to be like:*

* 27. 6.

a 18. 6. b *ſch.* 45. 1. c 25. 6.

Biſect AB in E; upon EB [a] make the parallelogram EG like to the parallelogram D; and [b] let EG = C + I. [c] Make the parallelogram NT = I, and like to the parallelogram given D, or EG; Draw the diameter FB; Make FO = KN, and FQ = KT; thorough O and Q draw the parallels SR, QZ. Then is the parallelogram AP that which was ſought.

For

For the Parallelograms D, EG, OQ, NT, ZR are all [d] like one to the other, and the Parallelogram EG = [e] NT + C = [e] OQ + C. [f] wherefore C = to the Gnomon OBQ [g] = AO + PG = [h] AO + EP = AP. Which was to be done.

[d] *Constr.* 24. 6.
[e] *Constr.*
[f] 3 *ax.*
[g] 2 *ax.*
[h] 43. 1.

PROP. XXIX.

Upon a right line given AB to apply a parallelogram AN equal to a right-lined figure given C, exceeding by a parallelogram OP, which shall be like to another parallelogram given D.

Biſect AB in E. Upon EB [a] make a Parallelogram EG like to D, which was given. And [b] let the Parallelogram HK = EG + C, and like to D given, or EG. Make FEL = [c] IH; and [c] FGM = IK. Thorough LM draw the parallels MN and RN, and AR parallel to NM, produce ABP, GBO. Draw the diameter FBN. Then is AN the parallelogram required.

[a] 18. 6.
[b] 25. 6.
[c] 3. 1.

For the parallelograms D, HK, LM, EG are [d] like. [e] Therefore the parallelogram OP is like to the parallelogram LM, or D. Alſo LM [f] = HK [f] = EG + C. [g] Therefore C = to the Gnomon ENG. But AL [h] = LB [k] = BM. [l] Therefore C = AN. Which was to be done.

[d] *conſtr.*
[e] 24. 6.
[f] *conſtr.*
[g] 3 *ax.*
[h] 36. 1.
[k] 43. 1.
[l] 2. & 1

PROP. XXX.

To cut a finite right line given AB according to extream and mean ratio (AB. AG :: AG. GB.)

[a] Cut AB in G, in ſuch wiſe that AB × BG = AGq. [b] Then BA. AG :: AG. GB. Which was to be done.

[a] 11. 2.
[b] 17. 6.

PROP. XXXI.

In right-angled triangles BAC, any figure BF described upon the side BC subtending the right-angle BAC, is equal to the figures BG, AL, described upon the sides BA, AC, containing the right-angle, like and alike situate to the former BF.

From the right angle BAC let down a perpendicular AD. Because that DC. CA :: [a] CA : CB, [b] therefore AL. BF :: DC. CB. Also, because DB. BA :: [a] BA. BC, [b] therefore BG. BF :: DB. EC. [c] Therefore AL + BG. BF :: DC + DB (BC). EC. [d] Therefore AL + BG = BF. Which was to be demonstrated.

Or thus: BG. BF :: [e] BAq. BCq. And [e] AL. BF :: ACq. BCq. [f] Therefore BG + AL. BF :: BAq + ACq. BCq. [g] Therefore whereas BAq + ACq = [h] BCq. [h] hence is BG + AL = BF. Which was to be demonstrated.

[a] *Cor.* 8. 6. [b] *Cor.* 20. 6. [c] 24. 5. [d] *Scb.* 14. 5. [e] 22. 6. [f] 24. 5. [g] *Scb.* 14. 5. [h] 47. 1.

Corollarium.

From this Proposition you may learn how to add or subtract any like figures, by the same method that is used in adding and subtracting of squares, in *Schol.* 47. 1.

PROP. XXXII.

If two triangles ABC, DCE having two sides proportional to two (AB. AC :: DC. DE) be so compounded or set together at one angle CD that their homologous sides be also parallel (AB to DC, and AC to DE,) then the remaining sides of those triangles (BC, CE) shall be found placed in one strait line.

For

For the angle A [a] = ACD [a] = D. and AB. AC [b] :: DC. DE. [c] therefore the angle B = DCE. Therefore the angle B + A [d] = ACE. But the angle B + A + ACB [e] = 2 right. [f] therefore the angle ACE + ACB = 2 right. [g] therefore BCE is a right line. Which was to be demonſtrated.

[a] 29. 1.
[b] *hyp.*
[c] 6. 6.
[d] 2 *ax.*
[e] 32. 1.
[f] 1 *ax.*
[g] 14. 1.

PROP. XXXIII.

In equal circles DBCA, HFGP, the angles BDC, FHG have the ſame ratio with their peripheries BC, FG on which they inſiſt; whether the angles be ſet at the centers (as BDC, FHG) or at the circumferences, A, E: And in like ſort are the Sectors BDC, FHG, becauſe deſcribed upon the centers.

Draw the right lines BC, FG. Make CI = CB, and GL = FG = LP, and joyn DI, HL, HP.

The arch BC [a] = CI, [a] alſo the arch FG, GL, LP are equal; [b] therefore the angle BDC = CDI, [b] and the angle FGH = GHL = LHP. Therefore the arch BI is as multiplex of the arch BC, as the angle BDI is of the angle BDC. And in like manner is the arch FP as multiplex of the arch FG, as the angle FHP is of the angle FHG. But if the arch BI ⊏, =, ⊐ FP, [c] then likewiſe is the angle BDI ⊏, =, ⊐ FHP. Therefore is the arch BC. FG :: [d] the angle BDC. FHG [e] :: $\frac{BDC}{2}$. $\frac{FHG}{2}$ [f] :: A. E. Which was to be demonſtrated.

[a] 28. 3.
[b] 27. 3.
[c] 27. 3.
[d] 6 *def.* 5.
[e] 15. 5.
[f] 20. 3.

Moreover, the angle BMC [g] = CNI; [h] and therefore the ſegment BCM = CIN. [k] Alſo the triangle BDC = CDI; [l] wherefore the Sector BDCM = CDIN. After the ſame manner

[g] 27. 3.
[h] 24. 3.
[k] 4. 1.
[l] 2 *ax.*

ner are the ſectors FHG, GHL, LHP equal one to the other. Therefore ſince accordingly as the arch BI ⊏, =, ⊐ FGP, ſo is likewiſe the ſector BDI ⊏, =, ⊐ EHP; [m] thence ſhall be the ſector BDC. FHG :: the arch BC. FG. Which was to be demonſtrated.

[m] 6 *def.* 5.

Corollaria.

11. 5. 1. Hence, *As ſector is to ſector, ſo is angle to angle.*

2. *The angle BDC in the center is to four right angles, as the arch BC, on which it inſiſts, to the whole circumference.*

For as the angle BDC is to a right angle, ſo is the arch BC to a quadrant. Therefore BDC is ſo to four right angles as the arch BC is to four quadrants, that is, the whole circumference. Alſo, the angle A. 2 right :: the arch BC. periphery.

3. Hence, *The arches IL, BC of unequal circles which ſubtend equal angles, whether at the centers, as IAL and BAC, or at the periphery, are like.*

For IL. periphery :: angle IAL (BAC). 4 right. Alſo arch BC. periphery :: angle BAC. 4 right. Therefore IL. periphery :: EC. periphery. And conſequently the arches IL and BC are like. Whence,

4. *Two ſemidiameters AB, AC cut off like arches IL, BC from concentrical peripheries.*

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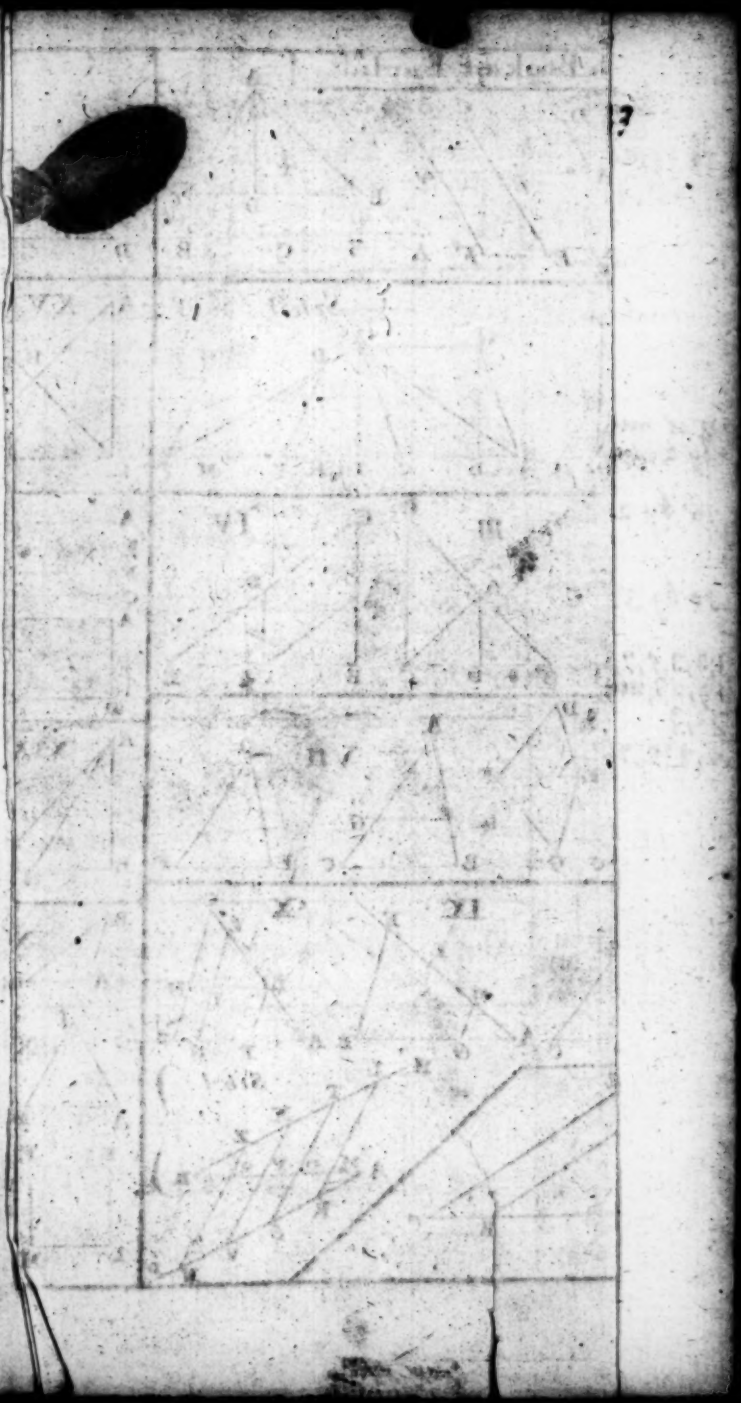

The Sixth Book of Euclid.

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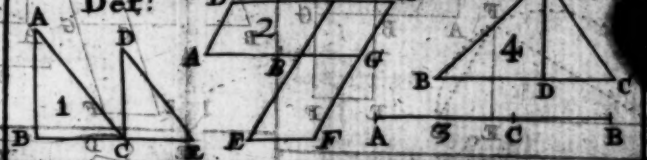

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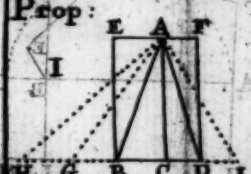

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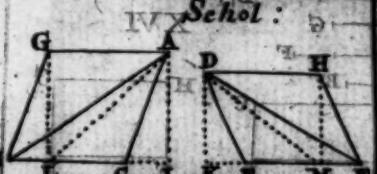

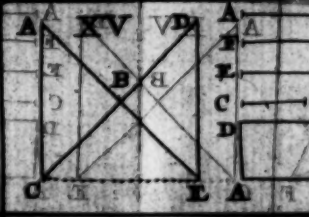

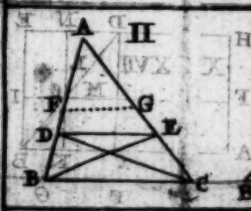

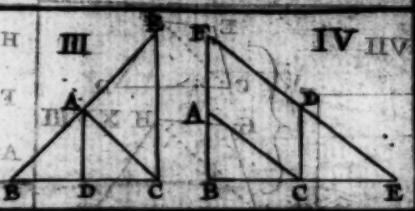

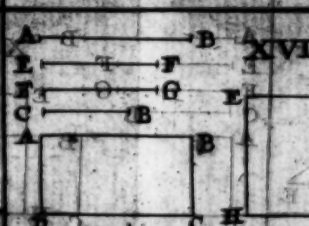

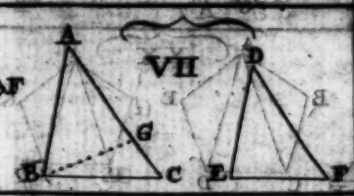

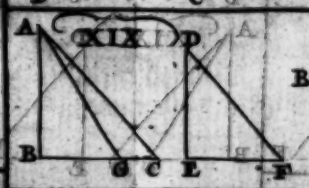

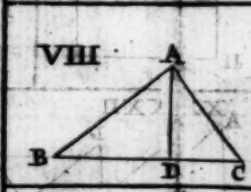

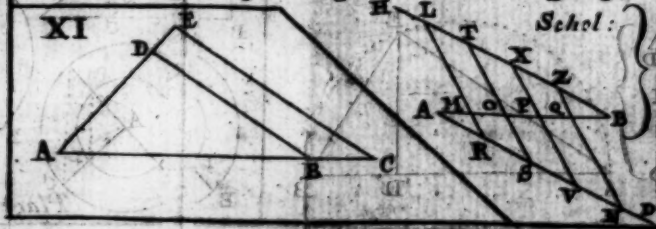

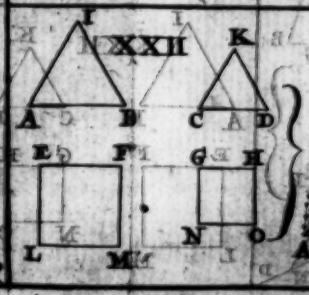

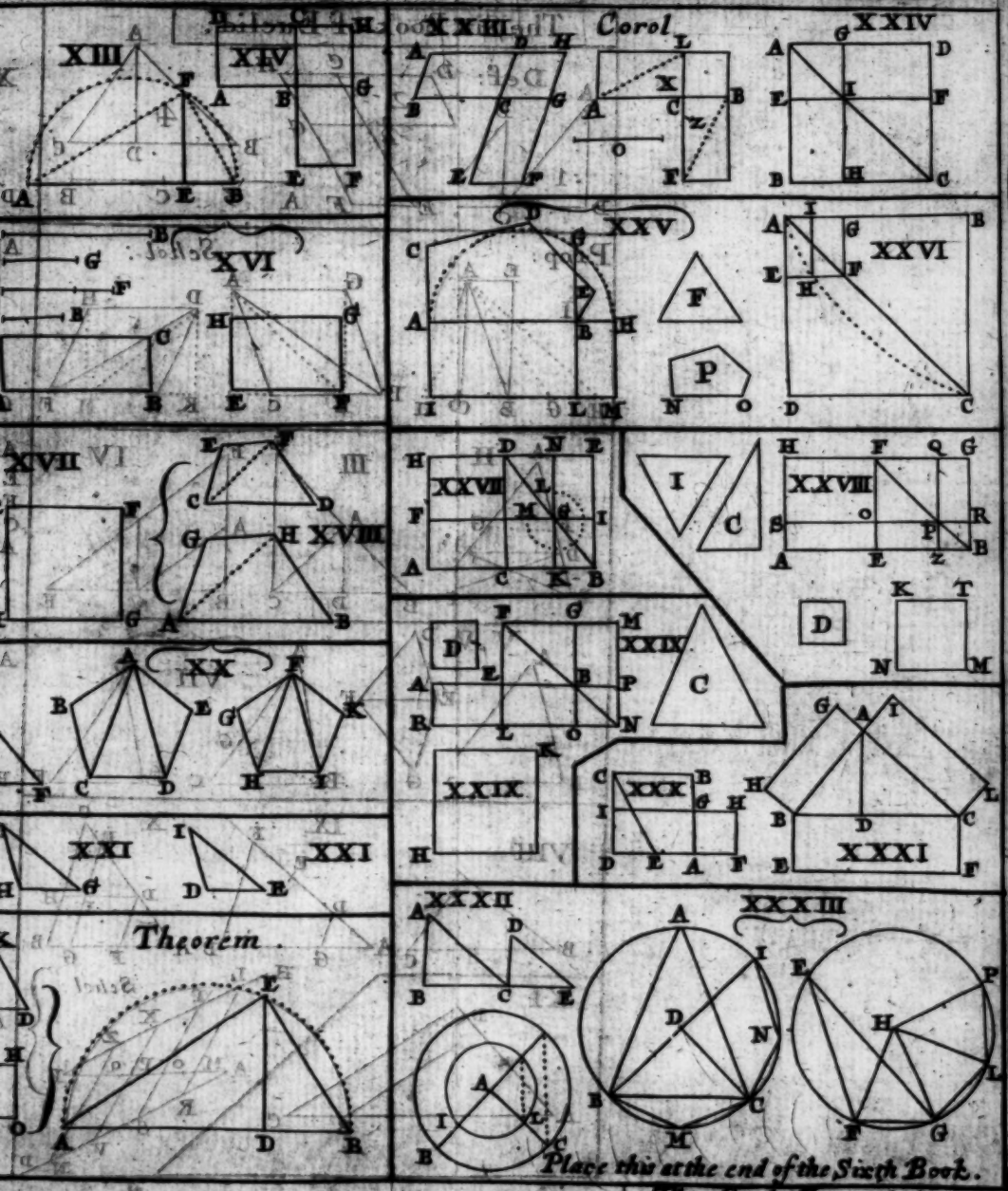
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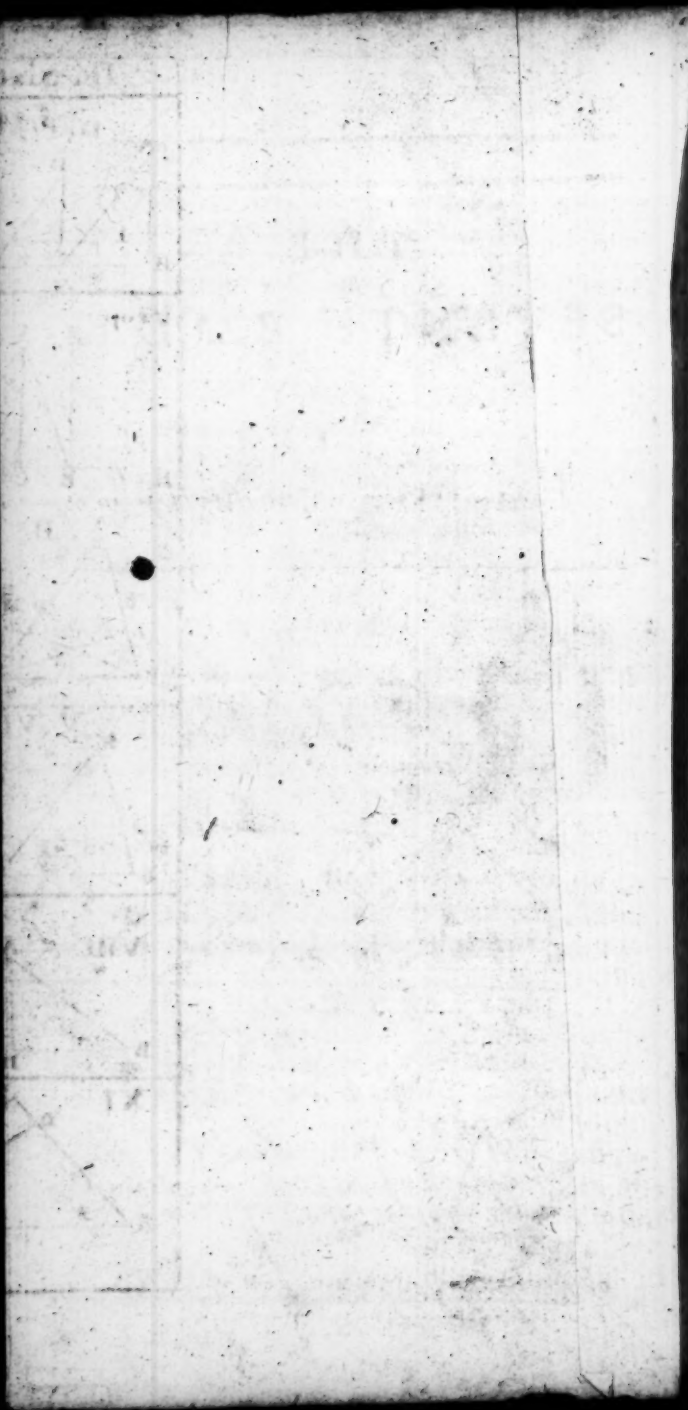


THE SEVENTH BOOK OF Euclid's *Elements*.

Definitions.

I. UNity is that, by which every thing that is, is called One.

II. Number is a multitude compoſed of unites.

III. One number is a Part of another, the leſſer of the greater, when the leſſer meaſureth the greater.

Every part is denominated from that number, by which it meaſures the number whereof it is a part; as 4 *is called the third part of* 12, *becauſe it meaſures* 12 *by* 3.

IV. But the leſſer number is termed Parts, when it meaſureth not the greater.

All parts whatſoever are denominated from thoſe two numbers, by which the greateſt common meaſure of the two numbers meaſures each of them; as 10 *is ſaid to be* $\frac{2}{3}$ *of the number* 15; *becauſe the greateſt common meaſure, which is* 5, *meaſures* 10 *by* 2, *and* 15 *by* 3.

12 is Multiplex of 2,3,4,6

V. A number is Multiplex (or Manifold) a greater in compariſon of a leſſer, when the leſſer meaſureth the greater.

VI. An even number is that which may be divided into two equal parts.

VII. But an odd number is that which cannot be divided into two equal parts; or, that which differeth from an even number by an unity.

8 is meaſured by 2,4.

VIII. A number evenly even is that which an even number meaſureth by an even number.

10 by 2.5

IX. But a number evenly odd is that which an even number meaſureth by an odd number.

15 by 3.5

X. A number oddly odd is that which an odd number meaſureth by an odd number.

1,2,3,5,7,11,13,17, 19,23,29, 31,3 2,3,5,7

XI. A Prime (or firſt) number is that, which is meaſured only by an unity.

XII. Numbers Prime the one to the other, are ſuch as onely an unity doth meaſure, being their common meaſure.

XIII. A Compoſed number is that which ſome certain number meaſureth.

XIV. Numbers Compoſed the one to the other, are they, which ſome number, being a common meaſure to them both, doth meaſure.

In this, and the preceding definition, unity is not a number.

XV. One number is ſaid to Multiply another, when the number multiplied is ſo often added to it ſelf, as there are unities in the number multiplying, and another number is produced.

Hence in every multiplication a unity is to the multiplier, as the multiplied is to the product.

Obſ.

Obſ. *That many times, when any numbers are to be multiplied (as A into B) the conjunction of the letters denotes the Product: So* $AB = A \times B$, *and* $CDE = C \times D \times E$.

XVI. When two numbers multiplying themſelves produce another, the number produced is called a Plane number; and the numbers which multiplied one another, are called the Sides of that number: *So* 2 (C) × 3 (D) = 6 = CD *is a plane number.*

XVII. But when three numbers multiplying one another produce any number, the number produced is termed a Solid number; and the numbers multiplying one another, are the ſides thereof: *So* 2 (C) × 3 (D) × 5 (E) = 30 = GDE *is a ſolid number.*

XVIII. A Square number is that which is equally equal; or, which is contained under two equal numbers. *Let A be the ſide of a Square; the Square is thus noted, AA, or Aq.*

XIX. A Cube is that number which is equally equal equally; or, which is contained under three equal numbers. *Let A be the ſide of a Cube; the Cube is thus noted, AAA, or Ac.*

In this definition, and the three foregoing, unity is a number.

XX. Numbers are proportional, when the firſt is as multiplex of the ſecond as the third is of the fourth; or, the ſame part; or when a part of the firſt number meaſures the ſecond, and the ſame part of the third meaſures the fourth, equally: and on the contrary. *So A. B :: C. D. that is,* 3. 9 :: 5. 15.

XXI. Like Plane, and Solid numbers are they, which have their ſides proportional: *Namely, not all the ſides, but ſome.*

XXII. A Perfect number is that which is equal to its own parts.

As 6 and 28. But a number that is less than its parts is called an Abounding *number; and a greater a* Diminutive: *So 12 is an abounding, 15 a diminutive number.*

XXIII. One number is said to measure another, by that number, which, when it multiplies, or is multiplied by it, it produceth.

In Division, a unity is to the quotient as the Divisor is to the Dividend. Note, *that a number placed under another with a line between them, signifies Division:* So $\frac{A}{B}$ = A divided by B, & $\frac{C \times A}{B}$ = C × A divided by B.

Two numbers are called Terms or Roots of Proportion, lesser than which cannot be found in the same proportion.

Postulates, or Petitions.

1. THat numbers equal or manifold to any number may be taken at pleasure.

2. That a greater number may be taken than any number whatsoever.

3. That Addition, Subtraction, Multiplication, Division, and the Extractions of Roots or Sides of Square and Cube numbers, be also granted as possible.

Axioms.

1. WHatsoever agrees with one of many equal numbers, agrees likewise with the rest.

2. Those parts that are the same to the same part, or parts, are the same amongst themselves.

3. Num-

3. Numbers that are the ſame parts of equal numbers, or of the ſame number, are equal amongſt themſelves.

4. Thoſe numbers, of whom the ſame number, or equal numbers, are the ſame parts, are equal amongſt themſelves.

5. Unity meaſures every number by the unities that are in it; that is, by the ſame number.

6. Every number meaſures it ſelf by a unity.

7. If one number, multiplying another, produce a third, the multiplier ſhall meaſure the product by the multiplied; and the multiplied ſhall meaſure the ſame by the multiplier.

Hence, no prime number is either a plane, ſolid, ſquare, or cube number.

8. If one number meaſure another, that number by which it meaſureth ſhall meaſure the ſame by the unities that are in the number meaſuring, that is, by the number it ſelf that meaſures.

9. If a number meaſuring another, multiply that by which it meaſureth, or be multiplied by it, it produceth the number which it meaſureth.

10. How many numbers ſo ever any number meaſureth, it likewiſe meaſures the number compoſed of them.

11. If a number meaſure any number, it alſo meaſureth every number which the ſaid number meaſureth.

12. A number that meaſures the whole and a part taken away, doth alſo meaſure the reſidue.

PROP.

PROP. I.

A.....E..G.B 8 5 3
C...F..D 1 1 2
H--- 3 2 1

If two unequal numbers AB, CD, being given, the leſſer CD be continually taken from the greater AB (and the reſidue EB from CD, &c.) by an alternate ſubtraction, and the number remaining do never meaſure the precedent, till the unity GB be taken; then are the numbers which were given AB, CD, prime the one to the other.

If you deny it, let AB, CD have a common meaſure, namely, the number H. Therefore [a] 11 *ax.* 7. H meaſuring CD, doth [a] alſo meaſure AE; and conſequently the remainder EB; [a] therefore it [b] 12 *ax.* 7. likewiſe meaſures CF, and [b] ſo the remainder FD; [a] wherefore it alſo meaſures EG. But it meaſured the whole EB, and [b] therefore it muſt meaſure that which remaineth GB, a unity, [c] 9 *ax.* 1. it ſelf being a number. [c] Which is abſurd.

PROP. II.

9 6
A..... E..... B 15 9 6
6 3
C..... F...D 2 1 1
G---

Two numbers AB, CD being given, not prime the one to the other, to find out their greateſt common meaſure FD.

Take the leſſer number CD from the greater AB as often as you can. If nothing remains, [a] 6. *ax.* 7. [a] it is manifeſt that CD is the greateſt common meaſure. But if there remains ſomething (as EB) then take it out of CD, and the reſidue FD out of EB, and ſo forward till ſome number

ber (FD) measure the said EB ([b] for this will be, before you come to a unity.) FD shall be the greatest common measure. [b] 1. 7.

For FD [c] measures EB, and [d] therefore also CF; and [e] consequently the whole CD; [d] therefore likewise AE; and so measures the whole AB. Wherefore it is evident that FD is a common measure. If you deny it to be the greatest, let there be a greater (G;) then whereas G measureth CD, it [d] must likewise measure AE, [e] and the residue EB, [d] as also CF, [e] and by consequence the residue FD, [g] the greater the less. [h] Which is absurd. [c] *const.* [d] 11. *ax.* 7. [e] 12. *ax.* 7. [d] 11. *ax.* 7. [e] 22. *ax.* 7. [g] *suppos.* [h] 9 *ax.* 1.

Corollarium.

Hence, A number that measures two numbers, does also measure their greatest common measure.

PROP. III.

A.....12
B.....8
D....4
C.....6
E..2
F---

Three numbers being given, A, B, C, not prime to one another, to find out their greatest common measure E.

Find out D the greatest common measure of the two numbers A, B. If D measures C the third, it is clear that D is the greatest common measure of all the three numbers. If D does not measure C, at least D and C will be composed the one to the other, by the Coroll. of the Proposition preceding. Therefore let E be the greatest common measure of the said nnmbers D and C, and it appears to be the number required.

For

PROP. I.

8 5 3
1 1 2
3 2 1

If two unequal numbers AB, CD, being given, the leſſer CD be continually taken from the greater AB (and the reſidue EB from CD, &c.) by an alternate ſubtraction, and the number remaining do never meaſure the precedent, till the unity GB be taken; then are the numbers which were given AB, CD, prime the one to the other.

If you deny it, let AB, CD have a common meaſure, namely, the number H. Therefore [a] H meaſuring CD, doth [a] alſo meaſure AE; and conſequently the remainder EB; [a] therefore it likewiſe meaſures CF, and [b] ſo the remainder FD; [a] wherefore it alſo meaſures EG. But it meaſured the whole EB, and [b] therefore it muſt meaſure that which remaineth GB, a unity, it ſelf being a number. [c] Which is abſurd.

[a] 11 *ax.* 7.
[b] 12 *ax.* 7.
[c] 9 *ax.* 1.

PROP. II.

15 9 6
2/6 4/6 3/6

Two numbers AB, CD being given, not prime the one to the other, to find out their greateſt common meaſure FD.

Take the leſſer number CD from the greater AB as often as you can. If nothing remains, [a] it is manifeſt that CD is the greateſt common meaſure. But if there remains ſomething (as EB) then take it out of CD, and the reſidue FD out of EB, and ſo forward till ſome number

[a] 6. *ax.* 7.

ber (FD) measure the said EB ([b] for this will be, before you come to a unity.) FD shall be the greatest common measure.

For FD [c] measures EB, and [d] therefore also CF; and [e] consequently the whole CD; [d] therefore likewise AE; and so measures the whole AB. Wherefore it is evident that FD is a common measure. If you deny it to be the greatest, let there be a greater (G;) then whereas G measureth CD, it [d] must likewise measure AE, [e] and the residue EB, [d] as also CF, [e] and by consequence the residue FD, [g] the greater the less. [h] Which is absurd.

[b] 1. 7.
[c] *const.*
[d] 11. *ax.* 7.
[e] 12. *ax.* 7.
[d] 11. *ax.* 7.
[e] 22. *ax.* 7.
[g] *suppos.*
[h] 9 *ax.* 1.

Corollarium.

Hence, A number that measures two numbers, does also measure their greatest common measure.

PROP. III.

A.....12
B.....8
D....4
C.....6
E..2
F---

Three numbers being given, A, B, C, not prime to one another, to find out their greatest common measure E.

Find out D the greatest common measure of the two numbers A, B. If D measures C the third, it is clear that D is the greatest common measure of all the three numbers. If D does not measure C, at least D and C will be composed the one to the other, by the Coroll. of the Proposition preceding. Therefore let E be the greatest common measure of the said nnmbers D and C, and it appears to be the number required.

For

[a] *conſt.* For E [a] meaſures C and D, and D meaſures
[b] 11. *ax.* 7. A and B; therefore [b] E meaſures each of the numbers A, B, C: neither ſhall any greater
[c] *cor.* 1. 7. (F) meaſure them; for if you affirm that, [c] then F meaſuring A and B, does likewiſe meaſure D their greateſt common meaſure; and in like manner, F meaſuring D and C, does alſo meaſure E [c] their greateſt common meaſure, [d] the greater the leſs. [e] Which is abſurd.
[d] *ſuppoſ.*
[e] 9 *ax.* 1.

Corollarium.

Hence, A number that meaſures three numbers, does alſo meaſure their greateſt common meaſure.

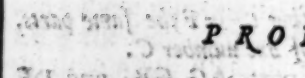
PROP. IV.

A.....6
B.....7 ——— *Every leſs number A is of every*
B.....18 *greater B either a*
B.....9 *part or parts.*

If A and B be
[a] 4. *def.* 7. prime to one another, [a] A ſhall be as many parts of the number B, as there are unities in A (as $6 = \frac{6}{7}7$.) But if A meaſures B, it is
[b] 3. *def.* 7. [b] plain that A is a part of B (as $6 = \frac{1}{3}18$.) Laſtly, if A and B be otherwiſe compoſed to
[c] 4. *def.* 7. one another, [c] the greateſt common meaſure ſhall determine how many parts A does contain of B; as $6 = \frac{2}{3}9$.

PROP.

PROP. V.



If a number A be a part of a number BC, and another number D the same part of another number EF; then both the numbers together (A + D) shall be the same part of both the numbers together (BC + EF,) which one number A is of one number BC.

For if BC be resolved into its parts BG, GC, equal to A; and EF also into its parts EH, HF equal to D; [a] the number of parts in BC shall be equal to the number of parts in EF. Therefore since A + D [b] = BG + EH = GC + HF, thence A + D shall be as often in BC + EF, as A is in BC. Which was to be demonstrated.

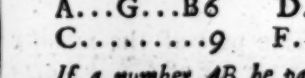
[a] *hyp.*

[b] *const. and 2. ax. 1.*

Or thus. Let $a = \frac{x}{2}$, and $b = \frac{y}{2}$, then [c] $2a = x$, and $2b = y$. wherefore $2a + 2b = x + y$. therefore $a + b = \frac{x+y}{2}$ Which was to be demonstrated.

[c] *2. ax. 1.*

PROP. VI.



If a number AB be parts of a number C, and another number DE the same parts of another number F; then both numbers together AB + DE shall be

be of both numbers together C + F the ſame parts, that one number AB is of one number C.

Divide AB into its parts AG, GB; and DE into its parts DH, HE. The multitude of parts in both AB, DE is equal by ſuppoſition; Wherefore ſince AG [a] is the ſame part of the number C, that DH is of the number F; therefore AG + DH [b] ſhall be the ſame part of the compounded number C + F, that one number AG is of one number C. [b] In like manner GB + HE is the ſame part of the ſaid C + F, that one number GB is of one number C. [c] Therefore AB + DE is the ſame parts of C + F, that AB is of C. Which was to be demonſtrated.

[a] *hyp.*
[b] 5. 7.
[c] 2. *ax.* 7.

Or thus. Let $a = \frac{3}{5}x$, and $b = \frac{3}{5}y$, and $x + y = g$. then becauſe $3a = 2x$, and $3b = 2y$, is $3a + 3b = 2x + 2y = 2g$. therefore $a + b = \frac{2}{3}g = \frac{2}{3} : x + y$.

PROP. VII.

5. 3
A.....E...B 8
6 10 6
G.....C.....F.....D 16

If a number AB be the ſame part of a number CD, that a part taken away AE is of a part taken away CF; then ſhall the reſidue EB be the ſame part of the reſidue FD that the whole AB is of the whole CD.

[a] 1. *poſt.* 7.
[b] 5. 7.
[c] 6. *ax.* 1.

[a] Let EB be the ſame part of the number GC that AB is of CD, or AE of CF. [b] therefore AE + EB is the ſame part of CF + GC that AE is of CF, or AB of CD. [c] therefore GF = CD. Take away CF common to both, and

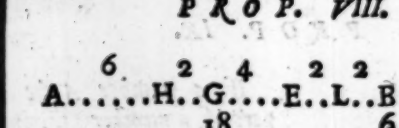
and [d] there remains GC = FD. [e] Wherefore EB is the ſame part of the reſidue FD (GC) that the whole AB is of the whole CB. Which was to be demonſtrated.

[d] 3. *ax.* 1.
[e] 2. *ax.* 7.

Or thus. Let a + b = x; and c + d = y; and x = 3 y, in like manner as a = 3 c; I ſay b = 3 d. For 3 c + 3 d [f] = 3 y = x [g] = a + b. take away from both 3 c [g] = a, and [h] there remains 3 d = b. Which was to be demonſtrated.

[f] 1. 2.
[g] *hyp.*

PROP. VIII.



If a number AB be the ſame parts of a number CD, that a part taken away AE is of a part taken away CF; the reſidue alſo EB ſhall be of the reſidue FD the ſame parts, that the whole AB is of the whole CD.

Divide AB into AG, GB, parts of the number CD; alſo AE into AH, HE, parts of the number CF; and take GL = AH = HE. [a] wherefore HG = EL. And becauſe [b] AG = GB, [c] therefore HG = LB. Now whereas the whole AG is the ſame part of the whole CD that the part taken away AH is of the part taken away CF, [d] the reſidue HG or EL ſhall be the ſame part alſo of the reſidue FD that AG is of CD. In like manner, becauſe GB is the ſame part of the whole CD, that HE or GL are of CF, [d] therefore the reſidue LB ſhall be the ſame part of the reſidue FD that GB is of the whole CD. Therefore EL + LB (EB) is the ſame parts of the reſidue FD, that the whole AB is of the

[a] 3. *ax.* 1.
[b] *conſt.*
[c] 3. *ax.* 1.
[d] 7. 7.

the whole CD. Which was to be demonſtrated.

Or thus more eaſily. Let $a+b=x$, and $c+d=y$. Alſo $y=\frac{3}{2}x$ as well as $c=\frac{3}{2}a$; or,
[e] 9 *ax.* 7. [e] which is the ſame, $3y=2x$; and $3c=2a$.
I ſay, $d=\frac{2}{3}b$. For $3c+3d$ [f] $=3y=2x$ [f]
[f] 1. 2. $=2a+2b$. [g] Therefore $3c+3d=2a+2b$.
[g] 1 *ax.* 1. take away from each $3c$ [h] $=2a$, and [k] there
[h] *hyp.* remains $3d=2b$. [l] therefore $d=\frac{2}{3}b$. Which
[k] 3 *ax.* 1. was to be demonſtrated.
[l] 8 *ax.* 7.

PROP. IX.

A....4		*If a number A be a part of a number BC, and another number D the ſame part of another number EF; then alternately what part or parts the firſt A is of the third D, the ſame part or parts ſhall the ſecond BC be of the fourth EF.*
4 4		
B....G....C 8		
D.....5		
5 5		
E.....H.....F 10		

A is ſuppoſed ⌐ D. therefore let BG, GC, and EH, HF, parts of the numbers BC, EF be equal; BG and GC to A; and EH, HF to D. The multitude of parts is put equal in both.
[a] 1 *ax.* 7. But it is clear that BG is [a] the ſame part of
and 4. 7. parts of EH, that GC is of HF; [b] wherefore
[b] 5. & 6. 7. BC (BG+GC) is the ſame part or parts of EF (EH+HF) that BG alone (A) is of EH alone (D.) Which was to be demonſtrated.

Or thus. Let $a=\frac{b}{3}$, and $c=\frac{d}{3}$; or $3a$
[*] 15. 5. $=b$, and $\frac{3}{a}c=d$, then is $\frac{c}{a}$ [*] $=\frac{3c}{3a}=\frac{d}{b}$.

PROP.

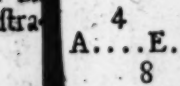
PROP. X.

If a number AB be parts of a number C, and another number DE the same parts of another number F; then alternately, what parts or part the first AB is of the third DE, the same parts or part shall the second C be of the fourth F.

AB is taken ⊐ DE, and C ⊐ F. Let AG, GB, and DH, HE be parts of the numbers C and F, *viz.* as many in AB, as in DE. It is manifest that AG is the same part of C, that DH is of F. [a] whence alternately AG is of DH, and likewise GB of HE, and [b] so conjointly AB of DE the same part, or parts, that C is of F. Which was to be demonstrated. [a] 9. 7. [b] 5. & 9 7.

Or thus. Let $a = 2\frac{b}{3}$, and $c = 2\frac{d}{3}$; or $3a = 2b$, and $3c = 2d$. Then is $\frac{c}{a} = \frac{3c}{3a} = \frac{2d}{2b} = \frac{d}{b}$.

PROP. XI.



If a part taken away AE be to a part taken away CF, as the whole AB is to the whole CD, the residue also EB shall be to the residue FD, as the whole AB is to the whole CD.

[a] 4. 7. [b] 20. *def.* 7. [c] 7. *ax.* 8. 7.

First, let AB be ⌐ CD: [a] then AB is either a part or parts of the number CD; and likewiſe AE is [b] the ſame part or parts of CF; [c] therefore the reſidue EB is the ſame part or parts of the reſidue FD that the whole AB is of the whole CD. [b] and ſo AB. CD :: EB. FD. But if AB be ⌐ CD, then according to what is already ſhewn, will CD. AB :: FD. EB. therefore by inverſion AB. CD :: EB, FD.

PROP. XII.

A, 4. C, 2. E, 3.
B, 8. D, 4. F, 6.

If there be divers numbers, how many ſoever, proportional (A. B :: C. D :: E. F;) then as one of the antecedents A is to one of the conſequents B, ſo ſhall all the antecedents (A + C + E) be to all the conſequents (B + D + F.)

[a] 20. *def* 7. [b] 5. & 6. 7. [c] 20. *def.* 7.

First, let A, C, E, be ⌐ B. D, F; then (becauſe of the ſame proportions) [a] ſhall A be the ſame part or parts of B that C is of D; [b] and likewiſe conjoyntly A + C ſhall be the ſame part or parts of B + D that A alone is of B alone. In the like manner A + C + E is the ſame part or parts of B + D + F that A is of B. [c] Therefore A + C + E. B + D + F :: A. B. But if A, C, E, be put greater than B, D, F, the ſame may be ſhewn by inverſion.

PROP. XIII.

A, 3. C, 4.
B, 9. D, 12.

If there be four numbers proportional (A. B :: C. D.) then alternately they ſhall alſo be proportional, (A. C :: B. D.)

First,

First, let A and C be ⌐ B and D, and A ⌐ C. By reaſon of the ſame proportion [a] ſhall A be the ſame part or parts of B, that C is of D. [b] Therefore alternately A is the ſame part or parts of C that B is of D; and ſo A . C :: B . D. But if A be ⌙ C, and A and C ſuppoſed ⌐ B and D, the matter will be the ſame by inverting the proportions.

[a] 20. *def.* 7.
[b] 9. *and* 10. 7.

PROP. XIV.

A, 9. D, 6. B, 6. E, 4. C, 3. F, 2.

If there be numbers, how many ſoever, A, B, C, and as many more equal to them in multitude, which may be compared two and two in the ſame proportion (A . B :: D . E, and B . C :: E . F;) they ſhall alſo, of equality, be in the ſame proportion (A . C :: D . F.)

For becauſe A . B :: D . E. [a] therefore alternately is A . D :: B . E :: [a] C . F. [a] and ſo again by changing A . C :: D . F. Which was to be demonſtrated.

[a] 13. 7.

PROP. XV.

1. D.. B...3. E.....6.

If a unite meaſure any number B, and another number D do equally meaſure ſome other number E; alternately alſo ſhall a unite meaſure the third number D, as often as the ſecond B doth the fourth E.

For ſeeing 1 is the ſame part of B, that D is of E; [a] therefore alternately ſhall 1 be the ſame part of D, that B is of E. Which was to be demonſtrated.

[a] 9. 7.

PROP. XVI.

B, 4. A, 3.
A, 3. B, 4.
AB, 12. BA, 12.

If two numbers A, B, multiplying themſelves the one into the other, produce any numbers AB, BA; the numbers produced AB and BA ſhall be equal the one to the other.

For becauſe AB = A × B, [a] therefore ſhall 1 be as often in A, as B in AB, [b] and by conſequence alternately 1 ſhall be as often in B as A in AB. But for that BA = B × A, [a] therefore ſhall 1 be as often in B, as A in BA; therefore as often as 1 is in AB, ſo often is 1 in BA; and [c] ſo AB = BA. Which was to be demonſtrated.

[a] 15. *def.* 7.
[b] 15. 7.
[c] 4. *ax.* 7.

PROP. XVII.

A, 3.
B, 2. C, 4.
AB, 6. AC, 12.

If a number A multiplying two numbers B, C, produce other numbers AB, AC; the numbers produc'd of them ſhall be in the ſame proportion that the numbers multiplyed are. (*AB. AC :: B. C.*)

For being AB = A × B, [a] therefore ſhall 1 be as often in A, as B in AB. [a] Likewiſe becauſe AC = A × C, therefore ſhall 1 be as often in A, as C in AC, and ſo alſo B as often in AB as C in AC; [b] wherefore B. AB :: C. AC; [c] and therefore alſo alternately B. C :: AB. AC. Which was to be demonſtrated.

[a] 15. *def.* 7.
[b] 20. *def.* 7.
[c] 13. 7.

PROP.

PROP. XVIII.

C, 5. C, 5. *If two numbers A, B,*
A, 3. B, 9. *multiplying any number C,*
AC, 15. BC, 45. *produce other numbers AC, BC; the numbers produced of them ſhall be in the ſame proportion that the numbers multiplying are (A. B :: AC. BC.)*

For [a] AC = CA, and BC [a] = CB; ſo the ſame C multiplying A and B produceth AC and BC; [b] therefore A. B :: AC. BC. Which was to be demonſtrated. [a] 16. 7. [b] 17. 7.

Scholium.

Hence is deduced the vulgar manner of reducing fractions ($\frac{1}{3}, \frac{7}{9}$) to the ſame denomination. For multiply 9 both by 3 and 5, and they produce $\frac{3}{27} = \frac{1}{3}$; becauſe by this, 3 . 5 :: 27 . 45. Likewiſe multiply 5 by 7 and 9; there ariſes $\frac{7}{45} = \frac{7}{9}$; becauſe 7 . 9 :: 35 . 45.

PROP. XIX.

A, 4. B, 6. C, 8. D, 12. *If there be four*
AD, 48. BC, 48. *numbers in proportion (A. B :: C. D) the number produced of the firſt and fourth (AD) is equal to the number which is produced of the ſecond and third (BC.) And if the number which is produced of the firſt and fourth (AD) be equal to that produced of the ſecond and third (BC) thoſe four numbers ſhall be in proportion (A. B :: C. D.)*

a 17. 7. b *hyp.* c 18. 7. d 9. 5. e *hyp.* f 7. 5. g 17. 7. h 18. 7. k 11. 5.

1. *Hypotheſis.* For AC. AD[a] :: C. D[b] :: A. B[c] :: AC. BC.[d] therefore AD = BC. Which was to be demonſtrated.

2. *Hypotheſis.* Becauſe[e] AD = BC, therefore AC. AD[f] :: AC. BC. But AC. AD[g] :: C. D. and AC. BC[h] :: A. B. [k] therefore C. D :: A. B. Which was to be demonſtrated.

PROP. XX.

A. 4. B. 6. C. 9. AC, 36. BB, 36. D, 6.

If there be three numbers in proportion (A. B :: B. C.) the number contained under the extremes (AC) is equal to the ſquare made of the middle (BB.) And if the number contained under the extremes be equal to that (BB.) produced of the middle, thoſe three numbers ſhall be in proportion $\left(\frac{A}{B} :: \frac{B}{C}.\right)$

a 1. *ax.* 7. b 19. 7. c *hyp.* d 19. 7.

1. *Hypotheſis.* For take D = B.[a] therefore A. B :: D (B). C. [b] wherefore AC = BD,[c] or BB. Which was to be demonſtrated.

2. *Hypotheſis.* Becauſe AC[c] = BD,[d] therefore A. B :: D (B). C. Which was to be demonſtrated.

PROP. XXI.

A ... G .. B 5. C .. H . D 3. I ..... 10 F ..... 6.

Numbers AB, CD, being the leaſt of all that have the ſame proportion with them (E, F) do equally meaſure the numbers E, F, having the ſame proportion with them; the greater AB the greater E, and the leſſer CD the leſſer F.

For

For AB.CD [a] :: E.F. [b] therefore alternately AB.E :: CD.F. [c] therefore AB is the same part or parts of E that CD is of F. Not parts; for if ſo, then let AG, GB be parts of the number E; and CH, HD, parts of the number F. [c] therefore AG.E :: CH.F. and by inverſion AG.CH [d] :: E.F [e] :: AB. CD. therefore AB, CD are not the leaſt in their proportion; which is contrary to the hypotheſis. Therefore, &c.

[a] *hyp.* [b] 13. 7. [c] 20. *def.* 7. [d] 13. 7. [e] *hyp.*

PROP. XXII.

A, 4. B, 3. C, 2. D, 12. E, 8. F, 6.

If there be three numbers A, B, C; and other numbers equal to them in multitude, D, E, F; which may be compared two and two in the ſame proportion: and if alſo the proportion of them be perturbed (A.B :: E.F. and B.C :: D.E.) then of equality they ſhall be in the ſame proportion (A.C :: D.F.)

For becauſe A.B [a] :: E.F, therefore ſhall AF = BE; and becauſe B.C :: [a] D.E, [b] therefore BE = CD, [c] and conſequently AF = CD. [d] Wherefore A.C :: D.F. Which was to be demonſtrated.

[a] *hyp.* [b] 19. 7. [c] 1. *ax.* 1. [d] 19. 7.

PROP. XXIII.

A. 9. B. 4. C. - - - D. - - - E. -

Numbers prime the one to the other, A, B, are the leaſt of all numbers that have the ſame proportion with them.

If it be poſſible, let C and D be leſs than A and B, and in the ſame proportion; [a] therefore C meaſures A equally as D meaſures B, namely, by the ſame number E; and ſo C ſhall

[a] 21. 7.

be [b] as often in A as 1 is in E; [c] likewiſe alternately, E as often in A as 1 in C. By the like inference as many times as 1 is in D, ſo many times ſhall E be in B. Therefore E meaſures both A and B; which conſequently are not prime the one to the other, contrary to the Hypotheſis.

[b] 23. *def.* 7.
[c] 15. 7.

PROP. XXIV.

A, 9. B, 4. C--- D--- E-.-

Numbers A, B, being the leaſt of all that have the ſame proportion with them, are prime the one to the other.

If it be poſſible, let A and B have a common meaſure C; and let the ſame meaſure A by D, and B by E; [a] therefore CD=A, [b] and CE=B. [b] Wherefore A.B :: D.E. But D and E are leſſer than A and B, as being but parts of them. Therefore A and B are not the leaſt in their proportion, againſt the Hypotheſis.

[a] 9. *ax.* 7.
[b] 17. 7.

PROP. XXV.

A, 9. B, 4. C, 3. D---

If two numbers A, B, be prime the one to the other, the number C meaſuring one of them A, ſhall be prime to the other number B.

For if you affirm any other D to meaſure the numbers B and C, [a] then D meaſuring C does alſo meaſure A; and conſequently A and B are not prime the one to the other. Which is againſt the Hypotheſis.

[a] 11. *ax.* 7.

PROP.

PROP. XXVI.

A, 5. B, 3. AB, 15. C, 8. E---- F----

If two numbers, A, B, be prime to any number C, the number also produced of them A B, shall be prime to the same C.

If it be possible, let the number E be a common measure to AB, and C; and let $\frac{AB}{E}$ be $= F$; [a] thence $AB = EF$; [b] wherefore also E. A :: B. F. But because A is prime to C, which E measures, [c] therefore E and A are prime to one another, [d] and so least in their own proportion, [e] and consequently they must measure B and F; namely, F shall measure B, and A shall measure F. Therefore seeing E measures both B and C, they shall not be prime to one another: contrary to the Hypothesis.

[a] 9. *ax.* 7.
[b] 19. 7.
[c] 25. 7.
[d] 23. 7.
[e] 21. 7.

PROP. XXVII.

A, 4. B, 5. Aq, 16. D, 4.

If two numbers A, B, be prime to one another, that also which is produc'd of one of them (Aq) shall be prime to the other B.

Take $D = A$; [a] therefore each of D, and A are prime to B. [b] wherefore AD or Aq is prime to B. Which was to be demonstrated.

[a] 1. *ax.* 7.
[b] 26. 7.

PROP. XXVIII.

A, 5. C, 4.
B, 3. D, 2.
AB, 15. CD, 8.

If two numbers A, B, be prime to two numbers C, D each to either of both, the numbers alſo produced of them AB, CD ſhall be prime to one another.

[a] 26. 7. For being A and B are prime to C, [a] therefore ſhall AB alſo be prime to the ſame. And by the ſame reaſon ſhall AB be prime to D.

[b] 26. 7. [b] Therefore AB is prime to CD. Which was to be demonſtrated.

PROP. XXIX.

A, 3. B, 2.
Aq, 9. Bq, 4.
Ac, 27. Bc, 8.

If two numbers A, B, be prime to one another, and each multiplying himſelf produce another number (Aq, and Bq;) then the numbers produced of them (Aq, Bq) ſhall be prime to one another. And if the numbers given at firſt, A, B, multiplying the ſaid produced numbers (Aq, Bq) produce others (Ac, Bc) thoſe numbers alſo ſhall be prime to one another: and this ſhall ever happen about the extremes.

[a] 27. 7. For becauſe A is prime to B, [a] therefore Aq ſhall be prime to B, and Aq being prime to B, [a] therefore Aq ſhall alſo be prime to Bq.

[b] 28. 7. Again, becauſe A is as well prime to B and Bq, as Aq is to the ſaid B and Bq, [b] therefore ſhall A × Aq, that is, Ac, be prime to B × Bq, that is, to Bc: And ſo forth of the reſt.

PROP.

PROP. XXX.

8 5
A.....B.....C 13. D---- *If two numbers AB, BC, be prime the one to the other; then both added together (AC) shall be prime to either of them AB, BC. And if both added together AC be prime to any one of them AB, the numbers also given in the beginning AB, BC, shall be prime to one another.*

1. *Hypothesis.* For if you would have AC, AB to be composed, let D be the common measure: [a] this shall measure the residue BC: and therefore AB, BC, are not prime to one another; which is against the Hypothesis. [a] 12. ax. 7.

2. *Hypothesis.* AC, AB being taken for prime to one another, let D be the common measure of AB, BC. [b] But seeing that measures the whole AC, therefore AC, AB, are not prime to one another; contrary to the Hypothesis. [b] 10. ax. 7

Corollarium.

Hence, A number, which being compounded of two, is prime to one of them, is also prime to the other.

PROP. XXXI.

A, 5. B, 8. *Every prime number A is prime to every number B, which it measureth not.*

For if any common measure doth measure both, A, B, [a] then A will not be a prime number; contrary to the Hypothesis. [a] 11. def. 7.

PROP.

PROP. XXXII.

A, 4. D, 3. B, 6. E, 8. AB, 24.

If two numbers A, B. multiplying one another, produce another AB, and some prime number D measure the number produced of them AB; then shall it also measure one of those numbers, A or B, which were given at the beginning.

Suppose the number D not to measure the number A, and let $\frac{AB}{D}$ be = E. [a] then AB = DE; [b] whence D. A :: B. E. [c] But D is prime to A; [d] therefore D and A are the least in their proportion; [e] and consequently D measures B as often as A measures E. Which was to be demonstrated.

[a] 9. *ax.* 7.
[b] 19. 7.
[c] *hyp. and* 31. 7.
[d] 23. 7.
[e] 21. 7.

PROP. XXXIII.

A, 12. B, 2.

Every composed number A is measured by some prime number B.

Let one or more numbers [a] measure A, of which let the least be B; that shall be a prime number: for if it be said to be composed, then some [a] lesser number shall measure it, [b] which shall also consequently measure A. Wherefore B is not the least of them which measure A. contrary to the Hypothesis.

[a] 13. *def.* 7.
[b] 11. *ax.* 7.

PROP. XXXIV.

A, 9.

Every number A, is either a prime, or measured by some prime number.

For A is necessarily either a prime or a composed number. If it be a prime, 'tis that we

we affirm. If compoſed, [a] then ſome prime number meaſureth it. Which was to be demonſtrated. [a] 33. 7.

PROP. XXXV.

A, 6. B, 4. C, 8.
D, 2. H--I--K----
E, 3. F, 2. G, 4. L---

How many numbers ſoever A, B, C, being given, to find the leaſt numbers E, F, G, that have the ſame proportion with them.

If A, B, C, be prime to one another, [a] they ſhall be the leaſt in their proportion. If they be compoſed, [b] let their greateſt common meaſure be D, which let meaſure them by E, F, G. Theſe are then leaſt in the proportion A, B, C. [a] 23. 7. [b] 3. 7.

For D × E, F, G, [c] produceth ABC, [d] therefore they are all in the ſame proportion. But allow other numbers H, I, K to be the leaſt in the ſame proportion; [e] which ſhall therefore equally meaſure A, B, C, namely, by the number L. [f] therefore L × H, I, K, ſhall produce A, B, C, [g] and conſequently ED = A = HL: [h] from whence E. H :: L. D. But E [k] ⊏ K; [l] therefore L ⊏ D, and ſo D is not the greateſt common meaſure of A, B, C. Which is againſt the Hypotheſis. [c] 9. *ax.* 7. [d] 17. 7. [e] 21. 7. [f] 9. *ax.* 7. [g] 1. *ax.* 1. [h] 19. 7. [k] *ſuppoſ.* [l] 20. *def.* 7.

Corollarium.

Hence, The greateſt common meaſure of how many numbers ſoever does meaſure them by the numbers which are leaſt of all that have the ſame proportion with them. Whereby appears the vulgar method of reducing Fractions to the leaſt terms.

PROP.

PROP. XXXVI.

Two numbers being given A, B, to find out the leaſt number which they meaſure.

A, 5. B, 4. AB, 20. D------ E---F---

1. *Caſe.* If A and B be prime the one to the other, AB is the number required. For it is manifeſt that A and B meaſure AB. If it be poſſible, let A and B meaſure ſome other number D ⌋ AB, [a] if you pleaſe by E, and F. [a] therefore AE = D = BF, [b] and ſo A.B :: F.E. But becauſe A and B [c] are prime the one to the other, [d] and ſo leaſt in their proportion, A ſhall [e] equally meaſure F as B does E. But B.E [f] :: AB.AE (D). [g] Therefore AB ſhall alſo meaſure D, which is leſs than it ſelf. Which is abſurd.

[a] 9.ax.7. & 1.ax.1.
[b] 19.7.
[c] *hyp.*
[d] 23.7.
[e] 21.7.
[f] 17.7.
[g] 20.*def*.7.

A, 6. B, 4. C, 3. D, 2. AD, 12. F---- G---H---

2. *Caſe.* But if A and B be compoſed one to another, [h] let there be found C and D the leaſt in the ſame proportion. [k] therefore AD = BC; and AD or BC ſhall be the number ſought for.

[h] 35.7.
[k] 19.7.

For it is [l] plain that B and D do meaſure AD or BC. Conceive A and B to meaſure F ⌋ AD, namely, A by G, and B by H. [m] therefore AG = F = BH, [n] whence A.B :: H.G [o] :: C.D. [p] and conſequently C equally meaſures H as D does G. But D.G [q] :: AD.AG (F.) therefore AD [r] meaſures F, the greater the leſs. Which is abſurd.

[l] 7.ax.7.
[m] 9.ax.7.
[n] 19.7.
[o] *conſt.*
[p] 21.7.
[q] 17.7.
[r] 20.*def*.7.

Corol-

Corollarium.

Hence, If two numbers multiply the leaſt that are in the ſame proportion, the greater the leſs, and the leſs the greater, the leaſt number which they meaſure ſhall be produced.

PROP. XXXVII.

A, 2. B, 3. E.....6 C----F---D

If two numbers A, B, meaſure any number CD, the leaſt number which they meaſure E ſhall alſo meaſure the ſame CD.

If you deny it, take E from CD as often as you can, and leave FD ⌐ E. therefore ſeeing A and B [a] meaſure E, [b] and E meaſures CF, [c] likewiſe A and B will meaſure CF. But [a] they meaſure the whole CD; [d] therefore alſo they meaſure the reſidue FD; and conſequently E is not the leaſt which A and B meaſure: Contrary to the Hypotheſis.

[a] *hyp.* [b] *conſtr.* [c] 11. *ax.* 7. [d] 12. *ax.* 7.

PROP. XXXVIII.

A, 3. B, 4. C, 6. D, 12.

Three numbers being given A, B, C, to find out the leaſt which they meaſure.

[a] Find D to be the leaſt that two of them A and B do meaſure; which if the third C do alſo meaſure, it is manifeſt that D is the number ſought for. But if C do not meaſure D, let E be the leaſt that C and D do meaſure, E ſhall be the number required.

[a] 36. 7.

A, 2. B, 3. C, 4. D, 6. E, 12. F---

For it appears by the 11. ax. 7. that A, B, C measure E; and it is easily shewn that they measure no other less F. For if you affirm they do, [b] then D measures F, [b] and consequently E measures the same F, the greater the less. Which is absurd.

[b] 37. 7.

Corollarium.

Hence it appears, that if three numbers measure any number, the least also, which they measure, shall measure the same.

PROP. XXXIX.

A, 12. B, 4. C, 3.

If any number B measure a number A, the number measured A, shall have a part C denominated of the number measuring B.

[a] *hyp.* For because $\frac{A}{B}$ [a] $= C$, [b] shall $A = BC$.

[b] 9. ax. 7.

[c] 7. ax. 7. [c] therefore $\frac{A}{C} = B$. Which was to be demonstrated.

PROP. XL.

A, 15. F, 3. C, 5.

If a number A have any part whatsoever B, the number C, from which the part B is denominated, shall measure the same.

[a] *hyp.* For being BC [a] $= A$, [b] thence $\frac{A}{C} = B$.

[b] 7. ax. 7. Which was to be demonstrated.

PROP.

PROP. XLI.

½ G, 12.
⅓ H---
¼

To find out a number G, which being the leaſt, containeth the parts given ½, ⅓, ¼.

[a] Let G be found the leaſt which the denominators 2, 3, 4, meaſure; [b] it is evident that G has the parts ½, ⅓, ¼. If it be poſſible, let H ⌐ G have the ſame parts; [c] therefore 2, 3, 4, meaſure H; and ſo G is not the leaſt which 2, 3, 4 meaſure: againſt the conſtruction.

[a] 38. 7.
[b] 39. 7.
[c] 40. 7.

The End of the Seventh Book.

THE

THE EIGHTH BOOK OF Euclid's *Elements*.

PROP. I.

A, 8. B, 12. C, 18. D, 27.
E-F--G---H----

IF there be divers numbers how many ſoever in continual proportion, A, B, C, D, and their extremes, A, D, prime to one another; then thoſe numbers A, B, C, D, are the leaſt of all numbers that have the ſame proportion with them.

For, if it be poſſible, let there be as many others E, F, G, H, leſs than A, B, C, D, and in the ſame proportion with them. [a] Therefore of equality A. D :: E. H. and conſequently A and D are prime numbers, [b] and ſo the leaſt in their proportion, [c] equally meaſuring E and H, which are leſs than themſelves. Which is abſurd.

[a] 14. 7.
[b] 23. 7.
[c] 21. 7.

PROP.

PROP. II.

1.

A, 2. B, 3.
Aq, 4. AB, 6. Bq, 9.
Ac, 8. AqB, 12. ABq, 18. Bc, 27.

To find out the leaſt numbers continually proportionals, as many as ſhall be required, in the proportion given of A to B.

Let A and B be the leaſt in the proportion given; Then Aq, AB, Bq, ſhall be the three leaſt in the ſame continual proportion that A is to B.

For AA. AB [a] :: A. B [a] :: AB. BB. Likewiſe becauſe A and B are [b] prime to one another, [c] ſhall Aq, Bq, be alſo prime to one another, [d] and ſo Aq, AB, Bq, are ∴ the leaſt in the proportion of A to B.

[a] 17. 7.
[b] 24. 7.
[c] 29. 7.
[d] 1. 8.

Moreover, I ſay Ac, AqB, ABq, Bc, are the four leaſt in the proportion of A to B. For AqA. AqB [e] :: A. B [e] :: ABA (AqB). ABB, [e] and A. B :: ABq. BBq (Ec.) Therefore ſince Ac, and Bc, are [f] prime to one another, likewiſe [g] ſhall Ac, AqB, ABq, Bc be the four leaſt ∴ in the proportion of A to B. In the ſame manner may you find out as many proportional numbers as you pleaſe. Which was to be done.

[e] 17. 7.
[f] 29. 7.
[g] 1. 8.

Corollaria.

1. Hence, If three numbers, being the leaſt, are proportional, their extremes ſhall be ſquares; if four, cubes. ~~of more~~ the powers of the Extreams will be always one leſs then the number

2. How

2. How many extremes proportional ſoever there be, being by this propoſition found to be the leaſt in the given proportion, they are prime to one another.

3. Two numbers, being the leaſt in the given proportion, do meaſure all the mean numbers whatſoever of the leaſt in the ſame proportion; becauſe they ariſe from the multiplication of them into certain other numbers.

4. Hence alſo it appears by the conſtruction, that the ſeries of numbers 1, A, Aq, AC; 1, B, Eq, Bc; Ac, AqB, ABq, BC, conſiſts of an equal multitude of numbers; and conſequently, the extream numbers of how many ſoever the leaſt continually proportionals are the laſt of as many other continually proportionals from a unite, as the extremes Ac, Bc, of the continually proportionals Ac, AqB, ABq, Bc, are the laſt of as many proportionals from a unite. 1, A, Aq, Ac; and 1, B, Bq, Bc.

5. 1, A, Aq, Ac; and B, BA, BAq; and Bq, ABq. are $\frac{\cdot}{\cdot\cdot}$ in the proportion of 1 to A. Alſo B, Bq, Bc; and A, AB, ABq; and Aq, AqB are $\frac{\cdot}{\cdot\cdot}$ in the proportion of 1 to B.

PROP. III.

A, 8. B, 12. C, 18. D, 28. *If there be numbers continually proportional, how many ſoever, A, B, C, D, being alſo the leaſt of all that have the ſame proportion with them; their extreams A, D, are prime to one another.*

[a] 2. 8. For if there be [a] found as many numbers the leaſt in the proportion of A to B, they ſhall be no other than A, B, C, D; therefore, by the ſecond *Coroll.* of the precedent *Prop.* the extreams

A

A and D are prime to one another. Which was to be demonſtrated.

PROP. IV.

A, 6. B, 5. C, 4. D, 3.
H, 4. F, 24. E, 20. G, 15.
I--K--L---

Proportions how many ſoever being given in the leaſt numbers (A to B, and C to D) to find out the leaſt numbers continually proportional in the proportions given.

[a] Find out E the leaſt number which B and C do meaſure; and let B meaſure E [b] as often as A does another F, *viz.* by the ſame number H. [b] Alſo let C meaſure the ſaid E as often as D meaſures another G. then F, E, G, ſhall be the leaſt in the proportions given. For AH [c] = F, and BC [c] = E; [d] therefore A.B :: AH.BH [e] :: F.E. In like manner C.D :: E.G; therefore F, E, G, are continually proportional in the proportions given. And they are moreover the leaſt in the ſaid proportions: for conceive other numbers I, K, L, to be the leaſt; [f] then A and B muſt equally meaſure I and K, [f] and C and D likewiſe K and L; and ſo B and C meaſure the ſame K. [g] Wherefore alſo E meaſures the ſame number K, which is leſs than it ſelf. Which is abſurd.

[a] 36. 7.
[b] 3. *poſt.* 7.
[c] 9. *ax.* 7.
[d] 8. 7.
[e] 7. 5.
[f] 21. 7.
[g] 37. 7.

A, 6. B, 5. C, 4. D, 3. E, 5. F, 7.
H, 24. G, 20. I, 15. K, 21.

But three proportions being given, A to B, C to D, and E to F; find out as before three numbers H, G, I, the leaſt continually in the proportions of A to B, and C to D. Then if

[h] 3. *post.* 7. if E measures I, [h] take another number K which may be equally measur'd by F; and those four numbers H, G, I, K, shall be continually the least in the given proportions; which we need go no other way to prove than we did in the first part.

A, 6. B, 5. C, 4. D, 3. E, 2. F, 7.
H, 24. G, 20. I, 15.
M, 48. L, 40. K, 30. N, 105.

If E do not measure I, let K be the least which E and I do measure; and as often as I measures K, let G as often measure L, and H also M. So likewise let F measure N as often as E measures K. The four numbers M, L, K, N, shall be least continually in the given proportions; which we may demonstrate as before.

PROP. V.

C, 4. E, 3.
D, 6. F, 16.
CD, 24. EF, 48.

ED, 18.

Plane numbers CD, EF, are in that proportion to one another which is compos'd of their sides.

[a] 17. 7.
[b] 20. *def.* 5.
[c] 11. 5.

For because CD. ED [a] :: C. E; [a] and ED. EF :: D. F. and $\frac{CD}{EF}$ [b] $= \frac{CD}{ED} + \frac{ED}{EF}$ [c] then shall be the proportion $\frac{CD}{EF} = \frac{C}{E} + \frac{D}{F}$. Which was to be demonstrated.

PROP.

PROP. VI.

A, 16. B, 24. C, 36. D, 54. E, 81. F, 4. G, 6. H, 9.

If there be numbers continually proportional how many soever, A, B, C, D, E, and the first A do not measure the second B, neither shall any of the other measure any one of the rest.

Because A does not measure B, [a] neither shall any one measure that which next follows; being A. B :: B. C :: C. D, &c. [b] Take three numbers, F, G, H, the least in the proportion of A to B. therefore since A does not measure B, [a] neither shall F measure G. [c] therefore F is not a unite. But F and H are prime to one another; and so, [d] being of equality A. C :: F. H. and F does not measure H, [a] neither shall A measure C; and consequently neither shall B measure D, nor C measure E, &c. because A. C [e] :: B. D [e] :: C. E, &c. In like manner four or five numbers being taken the least in the proportion of A to B, it will appear that A does not measure D and E; nor does B measure E and F, &c. Wherefore none of them shall measure any other. Which was to be demonstrated.

[a] 20. *def.* 7.
[b] 35. 7.
[c] 5. *ax.* 7.
[d] 3. 2.
[e] 14. 7.

PROP. VII.

A, 3. B, 6. C, 12. D, 24. E, 48.

If there be numbers continually proportional how many soever A, B, C, D, E, and the first A measure the last E, it shall also measure the second B.

If you deny that A measures B, [a] then neither shall it measure E; Which is contrary to the Hypothesis.

[a] 6. 7.

PROP.

PROP. VIII.

A,24. C,36. D,54. B,81. G,8. H,12. I,18. K,27. E,32. L,48. M,72. F,108.

If between two numbers A, B, there fall mean proportional numbers in continual proportion C, D; as many mean continually proportional numbers as fall between them, so many also mean continually proportional numbers shall fall between two other numbers E, F, which have the same proportion with them. (L, M.)

[a] Take G, H, I, K, the least ∴ in the proportion of A to C; [b] of equality shall G.K :: A.B [c] :: E.F. But G and K [d] are prime to one another. [e] Wherefore G measures E as often as K does F. Let H measure L, and I likewise M by the same number. [f] therefore E, L, M, F are in such proportion as G, H, I, K, that is, as A, B, C, D. Which was to be demonstrated.

[a] 35. 7.
[b] 14. 7.
[c] *hyp.*
[d] 3. 8.
[e] 21. 7.
[f] *constr.*

PROP. IX.

1. E,2. F,3. G,4. H,6. I,9. A,8. C,12. D,18. B,27.

If two numbers A, B be prime to one another, and mean numbers in continual proportion C, D fall between them; as many mean numbers in continual proportion as fall between them, so many means also (E, G; and F, I) shall fall in continual proportion between either of them and a unity.

It is evident, that 1, E, G, A, and 1, F, I, B, are ∴, and as many as A, C, D, B. namely, by the 4. Coroll. 2. 8. Which was to be demonstrated.

PROP.

PROP. XI.

A,8. I,12. K,18. B,27. E,4. DF,6. G,9. D,2. F,3. 1.

If between two numbers A, B, and a unite, numbers continually proportional (E, D, and F, G,) do fall; how many mean numbers in continual proportion fall between either of them and a unite, so many means also shall fall in continual proportion between them, I, K.

Name E. DF, G, and A, DqF (I) DG (K) B are ∴ by 2. 8. Therefore, &c.

PROP. XI.

A,2. B,3. Aq,4. AB,6. Bq,9.

Between two ſquare numbers Aq, Bq, there is one mean proportional number AB: and the ſquare Aq to the ſquare Bq is in double proportion of that of the ſide A to the ſide B.

[a] It is manifeſt that Aq, AB, Bq, are ∴; [b] and conſequently alſo $\frac{Aq}{Bq} = \frac{A}{B}$ doubly. Which was to be demonſtrated.

[a] 17. 7.
[b] 10. *def.* 5.

PROP. XII.

Ac,27. AqB,36. ABq,48. Bc,64. A,3. B,4. Aq,9. AB,12. Bq,16.

Between two cube numbers, Ac, Bc, there are two mean proportional numbers AqB, ABq: and the cube Ac is to the cube Bc in treble proportion of that in which the ſide A is to the ſide B.

[a] 2. 8. [b] 10 *def.* 5. [a] For Ac, AqB, ABq, Bc, are $\frac{\cdot\cdot}{\cdot\cdot}$ in the proportion of A to B; [b] and therefore $\frac{Ac}{Bc} = \frac{A}{B}$ trebly. Which was to be demonstrated.

PROP. XIII.

A, 2. B, 4. C, 8.
Aq, 4. AB, 8. Bq, 16. BC, 32. Cq, 64.
Ac, 8. AqB, 16. ABq, 32. Bc, 64. BqC, 128. BCq, 256. Cc, 512.

If there be numbers in continual proportion how many soever, A, B, C; and every of them multiplying it self produce certain numbers, the numbers produced of them Aq, Bq, Cq, shall be proportional: And if the numbers first given A, B, C, multiplying their products Aq, Bq, Cq, produce other numbers, Ac, Bc, Cc, they also shall be proportional; and this shall ever happen to the extremes.

[a] 2. 1. [b] 14. 7. For Aq, AB, Bq, BC, Cq [a] are $\frac{\cdot\cdot}{\cdot\cdot}$; [b] therefore of equality Aq. Eq :: Bq. Cq. Which was to be demonstrated.

[a] Also Ac, AqB, ABq, Bc, Bqc, BCq, Cc, are $\frac{\cdot\cdot}{\cdot\cdot}$; [b] therefore likewise of equality Ac. Bc :: Bc. Cc. Which was to be demonstrated.

PROP. XIV.

Aq, 4. AB, 12. Bq, 36.
A, 2. B, 6.

If a square number Aq measure a square number Bq, the side also of the one (A) shall measure the side of the other (B:) and if the side of one square A measure the side of another B, the square Aq shall likewise measure the square Bq.

1. Hypo-

1. *Hypotheſis.* For Aq . AB [a] :: AB . Bq; therefore ſeeing by the Hypotheſis Aq meaſures Bq, [b] it ſhall meaſure alſo AB. But Aq. AB :: A.B; [c] therefore alſo A meaſures B. Which was to be demonſtrated. [a] 2. & 11. 8. [b] 7. 8.

2. *Hypotheſis.* A meaſures B; [c] therefore Aq ſhall as well meaſure AB, [c] as AB meaſures Bq; [d] and conſequently Aq meaſures Bq. Which was to be demonſtrated. [c] 20. *def* 7. [d] 11. *ax.* 7.

PROP. XV.

A, 2. B, 6. Ac, 8. AqB, 24. ABq, 72. Bc, 216.

If a cube number AC meaſure a cube number Bc, then the ſide of the one (A) ſhall meaſure the ſide of the other (B:) And if the ſide A of one cube Ac meaſure the ſide B of the other BC, alſo the cube Ac ſhall meaſure the cube Bc.

1. *Hypotheſis.* For Ac, AqB, ABq, Bc [a] are ∴; therefore Ac, [b] meaſuring the extream Bc, ſhall alſo [c] meaſure the ſecond AqB. But Ac. AqB :: A. B; [d] therefore A ſhall alſo meaſure B. [a] 2. & 12. 8. [b] *hyp.* [c] 7. 8. [d] 20. *def.* 7.

2. *Hypotheſis.* A meaſures B; [d] therefore Ac meaſures AqB, which alſo meaſures ABq, and that Bc; [e] therefore Ac ſhall meaſure Bc. Which was to be demonſtrated. [e] 11. *ax.* 7.

PROP. XVI.

A, 4. B, 9. Aq, 16. Bq, 81.

If a ſquare number Aq do not meaſure a ſquare number Bq, neither ſhall the ſide of the one A meaſure the ſide of the other B: And if A the ſide of the one ſquare Aq do not meaſure B the ſide of the other Bq, neither ſhall the ſquare Aq meaſure the ſquare Bq.

1. *Hypothesis.* For if you affirm that A measures B, [a] then Aq also shall measure Bq. A. gainst the Hypothesis.

[a] 14. 8.

2. *Hypothesis.* If you maintain Aq to measure Bq, [a] then likewise A shall measure B. Contrary to the Hypothesis.

PROP. XVII.

A, 2. B, 3. Ac, 8. Ec, 27.

If a cube number Ac do not measure a cube number Bc, neither shall the side of one A, measure the side of the other B: And if A the side of one cube Ac do not measure B the side of the other Bc, neither shall the cube Ac measure the cube Bc.

[a] 15. 8.

1. *Hypothesis.* Let A measure B; [a] then Ac shall measure Bc; against the Hypothesis.

2. *Hypothesis.* Let Ac measure Bc; then A shall measure B: Which is also against the Hypothesis.

PROP. XVIII.

C, 6. D, 2. CD, 12. E, 9. F, 3. DE, 18. EF, 27.

Between two like plane numbers CD and EF there is one mean proportional number DE: And the plane CD is to the plane EF. in double proportion of that which the side C hath to the homologous side (or of like proportion) E.

[*] 21. *def.* 7. [a] 17. 7. [b] 11. 5. [c] 10. *def.* 5.

Being * by the Hypothesis C.D :: E.F; therefore by inversion C.E :: D.F. But C.E [a] :: CD.DE; [a] and D.F :: DE.EF; [b] therefore CD.DE :: DE.EF. [c] Wherefore the proportion of CD to EF is double to that of CD to DE, that is, to the proportion of C to E, or D to F.

Corol-

Corollarium.

Hence it is apparent, That between two like plane numbers there falls one mean proportional in the proportion of the homologous ſides.

PROP. XIX.

CDE, 30. DEF, 60. FGE, 120. FGH, 240.
CD, 6. DF, 12. FG, 24.
C, 1. D, 3. E, 5. F, 4. G, 6. H, 10.

Between two like ſolid numbers CDE, FGH, there are two mean proportional numbers DFE, FGE. And the ſolid CDE is to the ſolid FGH, in treble proportion of that which the homologous ſide C has to the homologous ſide F.

Whereas by the * Hypotheſis C.D :: F.G and D.E :: G.H; therefore [a] by inverſion ſhall C.F :: D.G :: [a] E.H. But CD.DF [b] :: G.F, and DF.FG [b] :: D.G; [c] wherefore CD. DF :: DF. FG :: E. H; [d] and accordingly CDE.DFE :: DFE.FGE :: E.H :: FGE.FGH. Therefore between CDE, FGH, fall two mean proportionals DFE, FGE. [e] And ſo it is plain that the proportion of CDE to FGH is treble to that of CDE to DFE, or C to F. Which was to be demonſtrated.

* 21. *def.* 7.
[a] 13. 7.
[b] 1. 7.
[c] 11. 5.
[d] 17. 7.
[e] 10. *def.* 5.

Corollarium.

Hereby it is manifeſt, That between two like ſolid numbers there fall two mean proportionals in the proportion of the homologous ſides.

PROP. XX.

A, 12. C, 18. B, 27. D, 2. E, 3. F, 6. G, 9. *If between two numbers A, B, there fall one mean proportional number C; those numbers A, B, are like plane numbers.*

[a] 35. 7. [a] Take D and E the least in the proportion of A to C, or C to B; then D measures A equally as E does C, *viz.* by the same number F; [b] 21. 7. [b] also D equally measures C, as E does B, [c] 9. *ax.* 7. *viz.* by the same number G. [c] Therefore [d] 16. *def.* 7. DF = A, and EG = B; [d] and consequently A and B are plane numbers. But because [e] 19. 7. EF [c] = C [c] = DG, [e] shall D. E :: F. G; and [f] 21. *def.* 7. alternately D. F :: E. G. [f] Therefore the plane numbers A and B are also like. Which was to be demonstrated.

PROP. XXI.

A, 16. C, 24. D, 36. B, 54. E, 4. F, 6. G, 9. H, 2. P, 2. M, 4. K, 3. L, 3. N, 6. *If between two numbers A, B, there fall two mean proportional numbers C, D; those numbers A, B, are like solid numbers.*

[a] 2. 8. [a] Take E, F, G, the least $\frac{\cdot\cdot}{\cdot\cdot}$ in the proportion of A to C; [b] 20. 8. [b] then E and G are like plane numbers: Let the sides of this be H and P, [c] 21. *def.* 7. and of that K and L; [c] therefore H. K :: P. L [d] *cor.* 18. 8. :: [d] E. F. But E, F, G, do [e] equally measure [e] 21. 7. A, C, D, *viz.* by the same number M; and likewise the said numbers E, F, G, do equally measure the numbers C, D, B, *viz.* by the same [f] 9. *ax.* 7. number N. [f] Therefore A = EM = HPM,

[f] and

f and B = GN = KLN; g and so g A and B are solid numbers. But for that C f = FM, and D f = FN; therefore shall M.N h :: FM. FN k :: C.D l :: E.F :: H.K :: P.L; m wherefore A and B are like solid numbers. Which was to be demonstrated.

f 17. *def.* 7
h 17. 7.
k 7. 5.
l *constr.*
m 21. *def.* 7.

Lemma.

AE,	BF,	CG,	DH,	*If proportional numbers*
A,	B,	C,	D,	*A, B, C, D measure proportional numbers AE, BF, CG, DH by the numbers E, F, G, H, these numbers (E, F, G, H) shall be proportional.*
E,	F,	G,	H.	

For being AEDH a = BFCG, a and AD = EC, b thence will $\frac{AEDH}{AD} = \frac{BFCG}{BC}$ c that is, EH = FG. d Therefore E.F :: G.H. Which was to be demonstrated.

a 19. 7.
b 1. *ax.* 7.
c 9. *ax.* 7.

Corollarium.

Hence $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$. d For 1. B :: B. Bq, d and 1. A :: A. Aq; e therefore $\frac{1}{A}$. B :: $\frac{B}{A}$. $\frac{Bq}{Aq}$ d therefore $\frac{Bq}{Aq} = \frac{B}{A} \times \frac{B}{A}$. In like manner $\frac{B}{Ac} \times \frac{B}{Ac} = \frac{Bc}{Acc}$. And so of the rest.

d 15. *def.* 7.
e *lem. prec.*

PROP. XXII.

Aq, B, C. 4, 8, 16. *If three numbers Aq, B, C be continually proportional, and the first Aq a ſquare, the third C ſhall alſo be a ſquare.*

For becauſe Aq C [a] = Bq, [b] thence is C = $\frac{Bq}{Aq}$ [c] = Q. $\frac{B}{A}$. But it is plain that $\frac{B}{A}$ is a number, [d] becauſe $\frac{Bq}{Aq}$ or C is a number. Therefore if three, &c.

[a] 20. 7. [b] 7. *ax.* 7. [c] *cor. of the lem. prec.* [d] *hyp. and* 14. 8.

PROP. XXIII.

Ac, B, C, D. 8, 12, 18, 27. *If four numbers A, B, C, D be continually proportional, and the firſt of them A be a cube, the fourth alſo D ſhall be a cube.*

For becauſe AcD [a] = BC, [b] therefore D = $\frac{BC}{Ac}$ [c] = $\frac{B}{Ac}$ × C; that is, (becauſe AcC = [d] Bq, and [b] thence C = $\frac{Bq}{Ac}$) D = $\frac{B}{Ac}$ × $\frac{Bq}{Ac}$ [e] = $\frac{Bc}{Acc}$ [e] = C. $\frac{B}{Ac}$. But it is evident [e] that $\frac{B}{Ac}$ is a number, becauſe $\frac{Bc}{Acc}$ or D is ſuppoſed a number. Therefore if four numbers, &c.

[a] 19. 7. [b] 7. *ax.* 1. [c] *cor. of the prec. lem.* [d] 20. 7. [e] 15. 8.

PROP.

PROP. XXIV.

A, 16. 24. B, 36. C, 4. 6. D, 9. *If two numbers A, B, be in the ſame proportion one to another, that a ſquare number C is to a ſquare number D, and the firſt A be a ſquare number, the ſecond alſo B ſhall be a ſquare number.*

Between C and D being ſquare numbers, * and ſo between A and B having the ſame proportion, [a] falls one mean proportional. Therefore [b] being A is a ſquare number, [c] B and ſhall be a ſquare number. Which was to be demonſtrated.

* 8. 8. [a] 11. 8. [b] *by p.* [c] 22. 8.

Corollaria.

1. Hence, if there be two like numbers AB, CD (A. B :: C. D) and the firſt AB be a ſquare, the ſecond alſo CD ſhall be a ſquare.

* For AB. CD :: Aq. Cq. * 12 & 18. 8.

2. From hence it appears, That the proportion of any ſquare number to any other not ſquare cannot poſſibly be declared in two ſquare numbers. Whence it cannot be Q. Q :: 1. 2. nor 1. 5 :: Q. Q, &c.

PROP. XXV.

C, 64. 96. 144. D, 216. A, 8. 12. 18. B, 27. *If two numbers A, B, be in the ſame proportion one to another, that a cube number C is to a cube number D, the firſt of them A being a cube number, the ſecond B ſhall likewiſe be a cube number.*

[a] 12. 8. [b] 8. 8. [c] *hyp.* [d] 23. 8.

[a] Between the cube numbers C and D, [b] and ſo between A and B having the ſame proportion, fall two mean proportionals; therefore [c] becauſe A is a cube, [d] ſhall B be a cube alſo. Which was to be demonſtrated.

Corollaria.

1. Hence, if there be two numbers ABC, DEF (A.B :: D.E, and B,C :: E,F;) and the firſt ABC be a cube, the ſecond DEF ſhall be a cube alſo.

*12, & 19. 8.

* For ABC. DEF :: Ac. Dc.

2. It is perſpicuous from hence, That the proportion of any cube number to any other number not a cube cannot be found in two cube numbers.

PROP. XXVI.

A,20. C,30. B,45. D,4. E,6. F,9.

Like plane numbers A, B, are in the ſame proportion one to another, that a ſquare number is in to a ſquare number.

[a] 18. 8. [b] 2. 8. [c] 14. 7.

[a] Between A and B falls one mean proportional number C; [b] take three numbers D, E, F the leaſt $\frac{\cdot\cdot}{\cdot\cdot}$ in the proportion of A to C, the extremes D, F, [b] ſhall be ſquare numbers. But of equality A . B [c] :: D . F, therefore A . B :: Q . Q. Which was to be demonſtrated.

PROP.

PROP. XXVII.

A, 16. C, 24. D, 36. B, 54. E, 8. F, 12. G, 18. H, 27. *Like ſolid numbers A, B, are in the ſame proportion one to another, that a cube number is to a cube number.*

[a] Between A and B fall two mean proportional numbers, namely, C and D: [b] Take four numbers E, F, G, H the leaſt ∴ in the proportion of A to C; [b] the extremes E, H, are cube numbers. But A. B [c] :: E. H :: C. C. Which was to be demonſtrated. [a] 19. 8. [b] 2. 8. [c] 14. 7.

Scholia.

1. From hence is inferred, that no numbers in proportion ſuperparticular, ſuperbipartient, or double, or any other manifold proportion not denominated from a ſquare number, are like plane numbers. See *Clavius*

2. Likewiſe, that neither any two prime numbers, nor any two numbers prime one to another, not being ſquares, can be like plane numbers.

The end of the Eighth Book.

THE

THE NINTH BOOK OF Euclid's *Elements*.

PROP. I.

A, 6. B, 54.
Aq, 36. 108. AB, 324.

IF two like plane numbers A, B multiplying one another, produce a number AB, the number produced A B shall be a square number.

[a] 17. 7. [b] 18. 8. [c] 8. 8.

For A . B [a] :: Aq . AB; wherefore since one mean proportional [b] falls between A and B, [c] likewise one mean proportional number shall fall between Aq and AB: Therefore being the first Aq is a square number, [d] the third AB shall be a square number too. Which was to be demonstrated.

[d] 22. 8.

Or thus. Let a b, c d, be like plane numbers; namely, a . b :: c . d; [x] therefore a d = b c: and so likewise a b c d, or a d b c [y] = a d a d = Q: a d.

[x] 19. 7. [y] 1. ax. 7.

PROP.

PROP. II.

A, 6. B, 54. Aq, 36. AB, 324. *If two numbers A, B, multiplying one another, produce a ſquare number AB, theſe numbers A, B are like plane numbers.*

For A.B [a] :: Aq.AB; wherefore being between Aq, AB, [b] there falls one mean proportional number; [c] likewiſe one mean ſhall fall between A and B: [d] therefore A and B are like planes. Which was to be demonſtrated.

[a] 17. 7. [b] 11. 8. [c] 8. 8. [d] 20. 8.

PROP. III.

A, 2. Ac, 8. Acc, 64. *If a cube number Ac multiplying it ſelf produce a number Acc, the number produced Acc ſhall be a cube number.*

For 1, A [a] :: A.Aq [b] :: Aq.Ac; therefore between 1 and Ac fall two mean proportionals. But 1.Ac [a] :: Ac.Acc; [c] therefore between Ac and Acc, fall alſo two mean proportionals: And ſo by conſequence ſeeing Ac is a cube, [d] Acc ſhall be a cube alſo. Which was to be demonſtrated.

[a] 15. *def.* 7. [b] 17. 7. [c] 8. 8. [d] 23. 8.

Or thus; aaa (Ac) multiplyed into it ſelf makes aaaaaa (Acc;) this is a cube, whoſe ſide is aa.

PROP. IV.

Ac, 8. Ec, 27. Acc, 64. Ac, Ec, 216. *If a cube number Ac multiplying a cube number Ec produce a number AcEc, the produced number AcEc ſhall be a cube.*

For

[a] 17. 7. [b] 12. 8. [c] 8. 8. [d] 23. 8.

For Ac. Bc [a] :: Acc. AcBc. But between Ac and Bc [b] two mean proportional numbers fall; [c] therefore there fall as many between Acc and AcBc. So that whereas Acc is a cube number; [d] AcBc ſhall be ſuch alſo. Which was to be demonſtrated.

Or thus, AcBc = aaabbb (ababab) = C: ab.

PROP. V.

A, 8. B, 27. Acc, 64. AcB, 216.

If a cube number Ac multiplying a number B produce a cube number AcB, the number multiplyed B ſhall alſo be a cube.

[a] 17. 7. [b] 12. 8. [c] 8. 8. [d] 23. 8.

For Acc. AcB [a] :: Ac. B. But between Acc and AcB [b] fall two mean proportionals; [c] therefore alſo as many ſhall fall between Ac and B; whence Ac being a cube number, [d] B ſhall be a cube number too. Which was to be demonſtrated.

PROP. VI.

A, 8. Aq, 64. Ac, 512.

If a number A multiplying it ſelf produce a cube Aq, that number A it ſelf is a cube.

[a] hyp. [b] 19. def. 7. [c] 5. 9.

For becauſe Aq [a] is a cube, and AqA (Ac) [b] alſo a cube; therefore [c] ſhall A be a cube. Which was to be demonſtrated.

PROP.

PROP. VII.

A, 6. B, 11. AB, 66. D, 2. E, 3.

If a compoſed number A multiplying any number B, produce a number AB, the number produced AB ſhall be a ſolid number.

Being A is a compoſed number, [a] ſome other number D meaſures it, conceive by E; [b] therefore A = DE; [c] whence DEB = AB is a ſolid number. Which was to be demonſtrated.

[a] 13. *def.* 7. [b] 9. *ax.* 7. [c] 17. *def.* 7.

PROP. VIII.

1. a, 3. a^2, 9. a^3, 27. a^4, 81. a^5, 243. a^6, 729.

If from a unite there be numbers continually proportional how many ſoever (1, a, a^2, a^3, a^4, &c.) the third number from a unite a^2 is a ſquare number; and ſo are all forward, leaving one between (a^4, a^6, a^8, &c.) But the fourth a^3 is a cube number; and ſo are all forward, leaving two between (a^6, a^9, &c.) The ſeventh alſo a^6 is both a cube number and a ſquare; and ſo are all forward, leaving five between (a^{12}, a^{18}, &c.)

For 1. a^2 = Q. a. and a^4 = aaaa = Q. aa. and a^6 = aaaaaa = Q. aaa, &c.

2. a^3 = aaa = C. a. and a^6 = aaaaaa = C. aa. and aaaaaaaaa = C. aaa, &c.

3. a^6 = aaaaaa = C. aa = Q. aaa. therefore, &c.

Or according to *Euclid*: Becauſe 1. a [a] :: a. a^2. [b] ſhall a^2 = Q; and therefore ſeeing a^2, a^3, a^4, are ∴ [c] the third a^4 ſhall be a ſquare number; and ſo likewiſe a^6, a^8, &c. Alſo becauſe 1. a [a] :: a^2. a^3; therefore ſhall

[a] *hyp.* [b] 20. 7. [c] 12. 8.

a^3

b 20. 7. $a^3 b = a^2 \times a = C : a$. d therefore the fourth
d 23. 8. from a^3, namely, a^6, shall be likewise a cube, &c. and consequently a^6 is both a cube and a square number, &c.

PROP. IX.

1. a, 4. a^2, 16. a^3, 64. a^4, 256, &c. *If from*
1. a, 8. a^2, 64, a^3, 512. a^4, 4096. *a unite there be numbers how many soever continually proportional* (1; a, a^2, a^3, &c.) *and the number following the unite* (a) *be a square; then all the rest,* a^2, a^3, a^4, &c. *shall be squares too. But if the number next the unite* (a) *be a cube, then all the following numbers* a^2, a^3, a^4, &c. *shall be cube numbers.*

1. *Hypothesis.* For a^2, a^4, a^6, &c. are
a 22. 8. square numbers by the precedent *Prop.* also being a is taken to be a square, a therefore the third a^3 shall be a square, and likewise a^5, a^7, &c. and so all.

b 23. 8. 2. *Hypothesis.* a is taken to be a cube, b therefore a^4, a^7, a^{10} are cubes: but by the prec. a^3, a^6, a^9, &c. are cubes. Lastly, because
c 20. 7. 1. a :: a, a^2, c therefore shall $a^2 = Q : a$.
d 3. 9. But a cube multiplyed into it self d produces
e 23. 8. a cube; therefore a^2 is a cube, e and consequently the fourth from it a^3, and in like manner a^8, a^{11}, &c. are cubes; therefore all. Which was to be demonstrated.

Peradventure more clearly thus. Let b be the side of the square number a, and so the series a, a^2, a^3, a^4, &c. will be otherwise expressed, thus, bb, b^3, b^4, b^5, b^6, &c. It is evident that all these numbers are squares and may be thus expressed, Q : b, Q : bb, Q : bbb, Q : bbbb, &c.

In

In like manner, if b be the ſide of the cube a, the ſeries may be expreſſed thus, b^3, b^6, b^9, b^{12}, &c. or C: b, C: b^2, C: b^3, C: b^4, &c.

PROP. X.

1, a, a^2, a^3, a^4, a^5, a^6.
1, 2, 4, 8, 16, 32, 64.

If from a unite there be numbers how many ſoever continually proportional (1, a, a^2, a^3, &c.) and the number next the unite (a) be not a ſquare number, then is none of the reſt following a ſquare number, excepting a^2 the third from the unite, and ſo all forward, leaving one between (a^4, a^6, a^8, &c.) But if that (a) which is next after the unite, be not a cube number, neither is any other of the following numbers a cube, ſaving a^3 the fourth from the unite, and ſo all forward, leaving two between, a^6, a^9, a^{12}, &c.

1. *Hypotheſis.* For if it be poſſible, let a^5 be a ſquare number; therefore becauſe a. a^2 :: a^4. a^5, and by inverſion a^5. a^4 :: a^2. a, and alſo a^5 and a^4 [b] ſquare numbers, and the firſt a^2 a ſquare; [c] therefore a ſhall be likewiſe a ſquare; contrary to the Hypotheſis.

[a] *hyp.*
[b] *ſuppoſ.* & 8. 9.
[c] 24. 8.

2. *Hypotheſis.* If it may be, let a^4 be a cube; being [d] of equality a^4. a^6 :: a. a^3, and inverſely a^6. a^4 :: a^3. a; and alſo being a^6 and a^4 are cubes, and the firſt a^3 a cube, [e] therefore a ſhall be a cube alſo; againſt the Hypotheſis.

[d] 24. 7.
[e] 25. 8.

PROP.

PROP. XI.

1. a, a^2, a^3, a^4, a^5, a^6.
1, 3, 9, 27, 81, 243, 729.

If there be numbers how many soever in continual proportion from a unite (1, a, a^2, a^3, &c.) the leſs meaſureth the greater by ſome one of them that are amongſt the proportional numbers.

[a] 5. ax. 7. & 20. def. 7. [b] 14. 7.

Becauſe 1. a :: a. aa; [a] therefore $\frac{aa}{a} = a =$ $\frac{aaa}{aa}$. Alſo becauſe 1. aa [b] :: a. aaa; [a] therefore $\frac{aaa}{a} = aa = \frac{a^4}{aa} = \frac{a^5}{a^3}$, &c. Laſtly, becauſe 1. a^3 [b] :: a. a^4; therefore $\frac{a^4}{a} = a^3 = \frac{a^6}{a^3}$, &c.

Corollarium.

Hence, if a number that meaſures any one of proportional numbers, be not one of the ſaid numbers, neither ſhall the number by which it meaſures the ſaid proportional numbers, be one of them.

PROP. XII.

1, a, a^2, a^3, a^4.
1, 6, 36, 216, 1296.
B, 3.

If there be numbers how many ſoever in continual proportion from a unite (1, a, a^2, a^3, a^4,) whatſoever prime numbers B meaſure the laſt a^4, the ſame (B) ſhall alſo meaſure the number (a) which follows next after the unite.

If

If you ſay B does not meaſure a, [a] then B is prime to a; [b] and alſo B is prime to a^2; [c] and ſo conſequently to a^4, which it is ſuppoſed to meaſure. Which is abſurd.

[a] 31. 7.
[b] 27. 7.
[c] 26. 7.

Corollaria.

1. Therefore every prime number that meaſures the laſt, does alſo meaſure all thoſe other numbers that precede the laſt.

2. If any number not meaſuring that next to the unite, does yet meaſure the laſt, it is a compoſed number.

3. If the number next to the unite be a prime, no other prime number ſhall meaſure the laſt.

PROP. XIII.

1, a, a^2, a^3, a^4,
1, 5, 25, 125, 625.
H -- G -- F -- E --

If from a unite be numbers in continual proportion how many ſoever (a, a^2, a^3, &c.) and that after the unite (a) a prime; then ſhall no other meaſure the greateſt number, but thoſe which are amongſt the ſaid proportional numbers.

If it be poſſible, let ſome other E meaſure a^4, *viz.* by F; [a] then F ſhall be ſome other beſide a, a^2, a^3. But becauſe E meaſuring a^4, does not meaſure a, [b] therefore E ſhall be a compoſed number, [c] and ſome prime number meaſure it, [d] which does conſequently meaſure a^4, [e] and ſo is no other than a; therefore a meaſures E. So alſo may F be ſhewn to be a compoſed number, meaſuring a^4, and ſo that a meaſures F. Therefore ſeeing $EF^f = a^4 = a \times a^3$, [g] ſhall a. E :: F. a^3. Conſequently,

[a] *cor.* 11. 9.
[b] 2. *cor.* 12. 9.
[c] 33. 7.
[d] 11. *ax.* 7.
[e] 3. *cor.* 12. 9.
[f] 9. *ax.* 7.
[g] 19. 7.

[h] 20.*def.*7. quently, whereas a measures E, [h] likewise F shall equally measure a^3, *viz.* by the same
[k] *cor.*11.9. number G. [k] Nor shall G be a, or a^2; therefore, as before, G is a composed number, and a measures it. Wherefore being that $FG^{f} = a^3 = a^2 \times a$ g shall a. F :: G. a^2; and so because A measures F, [h] G shall equally measure a^2, *viz.* by the same number H, [k] which is not
[l] 23.*def.*7. a. Therefore being $GH = a^3 = aa$, [l] thence
[m] 20.*def.*7. H. a :: a. G; and because a measures G (as before) [m] H also shall measure a, which is a prime number. Which is impossible.

PROP. XIV.

If certain prime numbers
A, 30. *B, C, D, do measure the least*
B, 2. C, 3, D, 5. *number a, no other prime num-*
E--F--- *ber E shall measure the same, besides those that measured it at first.*

[a] 9.*ax.*7. If it is possible, let $\frac{A}{E} = F$; [a] then A
[b] 32.7. $= EF$; [b] therefore every of the prime numbers B, C, D measures one of those EF; not E, which is taken to be a prime; therefore F, which is less than it self A, contrary to the Hypothesis.

PROP. XV.

A, 9. B, 12. C, 16. *If three numbers conti-*
D, 3. E, 4. *nually proportional A, B, C, be the least of all that have the same proportion with them; any two of them added together shall be prime to the third.*

[a] Take

[a] Take D and E the least in the proportion of A to B; [b] then A = Dq, and [b] C = Eq, [c] and B = DE. But because D [c] is prime to E, [d] therefore shall D + E be prime to both D and E; [*] therefore D × D + E [e] = Dq + DE (= A + B) is prime to E, and so to C or Eq. Which was to be demonstrated.

[a] 35. 7.
[b] 2. 8.
[c] 24. 7.
[d] 30. 7.
[*] 26. 7.
[e] 3. 2.
[f] *before.*

[f] In like manner DE + Eq (B + C) is prime to D, and consequently to A = Dq. Which was to be demonstrated.

[f] 27. 7.

Lastly, because B [h] is prime to D + E, it shall also be prime to the square of it [k] Dq + 2 DE + Eq (A + 2 B + C;) [l] wherefore the said B shall be prime to A + B + C, [l] and so likewise to A + C. Which was to be demonstrated.

[h] 26. 7.
[k] 4. 2.
[l] 30. 7.

PROP. XVI.

A, 3. B, 5. C---. *If two numbers A, B, be prime to one another, the second B shall not be to any other C, as the first A is to the second B.*

If you affirm A. B :: B. C; then whereas A and B [a] are the least in their proportion, A [b] shall measure B as many times as B does C; but A [c] measures it self also; therefore A and B are not prime to one another, against the Hypothesis.

[a] 23. 7.
[b] 21. 7.
[c] 6. *ax.* 7.

PROP. XVII.

A, 8. B, 12. C, 18. D, 27. E---. *If there be numbers how many soever in continual proportion A, B, C, D, and the extremes of them A, D be prime one to another, the*

the last D shall not be to any other number E, as the first A is to the second B.

Suppose A. B :: D. E; then alternately A. D :: B. E: therefore seeing A and B are [a] the least in their proportion, A [b] shall measure B, [c] and B likewise C, and C the following number D; [d] and so A shall measure the said number D. Wherefore A and D are not prime to one another, contrary to the Hypothesis.

[a] 23. 7.
[b] 21. 7.
[c] 20. def. 7.
[d] 11. ax. 7.

PROP. XVIII.

A, 4. B, 6. C, 9. Bq, 36. *Two numbers being given A, B, to consider if there may be a third number found proportional to them C.*

If A measure Bq by any number C, [a] then AC = Bq; from whence [b] it is manifest that A. B :: B. C. Which was to be demonstrated.

[a] 9. ax. 7.
[b] 20. 7.

A, 6. B, 4. Bq, 16. But if A do not measure Bq, there will not be any third proportional. For suppose A. B :: B. C; [a] then AC = Bq, [c] and consequently $\frac{Bq}{A} = C$; namely, A measures Bq. Which is against the Hypothesis.

[a] 7. ax. 7.

PROP. XIX.

A, 8. B, 12. C, 18. D, 27. EC, 216. *Three numbers being given A, B, C, to consider if a fourth proportional to them D may be found.*

If A measures BC by any number D, [a] then AD = BC; [b] therefore it appears that A. B :: C. D. Which was required.

[a] 9. ax. 7.
[b] ax. 19. 7.

But

But if A do not measure BC, then there can no fourth proportional be found; which may be shewn as in the precedent Proposition.

PROP. XX.

A, 2. B, 3. C, 5. D 30. G----

More prime numbers may be given than any multitude whatsoever of prime numbers A, B, C, propounded.

[a] Let D be the least which A, B, C measure: If D + 1 be a prime, the case is plain; if composed, [b] then some prime number, conceive G, measures D + 1; which is none of the three A, B, C: For if it be, seeing it [c] measures the whole D + 1, [d] and the part taken away D, [e] it shall also measure the remaining unite. Which is absurd. Therefore the propounded number of prime numbers is increased by D + 1, or at least by G.

[a] 38. 7.
[b] 33. 7.
[c] *suppos.*
[d] *constr.*
[e] 12. *ax.* 7.

PROP. XXI.

A 5 E 5 B 3 F 3 C 2 G 2 D 20.

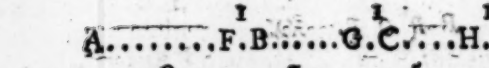
If even numbers, how many soever, AB, BC, CD, be added together, the whole AD shall be even.

[a] Take EB = ½ AB, and FC = ½ BC, and GD = ½ CD; [b] it is plain that EB + FC + GD = ½ AD: [c] therefore AD is an even number. Which was to be demonstrated.

[a] 6. *def.* 7.
[b] 12. 7.
[c] 6. *def.* 7.

PROP.

PROP. XXII.



If odd numbers, how many soever, AB, BC, CD, DE, be added together, and their multitudes even, the whole also AE shall be even.

A unite being taken from each odd number, there will [a] remain AF, BG, CH, DL, even numbers, [b] and thence the number compounded of them will be even; add to them the [c] even number made of the remaining unites, and the [d] whole AE will thereby be even. Which was to be demonstrated.

[a] 7. def. 7.
[b] 21. 9.
[c] hyp.
[d] 21. 9.

PROP. XXIII.

If odd numbers, how many soever, AB, BC, CD, be added together, and the multitude of them be odd, the whole AD shall be odd.

For CD one of the odd numbers being taken away, the number compounded of the others AC [a] is even. Whereto add CD — 1, [b] the whole AE is also even; wherefore the unite being restored, the whole AD [c] will be odd. Which was to be demonstrated.

[a] 22. 9.
[b] 21. 3.
[c] 7. def. 7.

PROP.

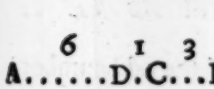
PROP. XXIV.

If an even number AB be taken away from an even number AC, that which remains BC ſhall be even.

For if BD (BC — 1) be odd, [a] BC (BD + 1) will be even. Which was to be demonſtrated. But if you ſay BD is even, becauſe AB [b] is even, [c] thence AD will be ſo; [a] and conſequently AC (AD + 1) will be odd, contrary to the Hypotheſis; therefore BC is even. Which was to be demonſtrated.

[a] 7. *def.* 7.
[b] *hyp.*
[c] 21. 9.

PROP. XXV.

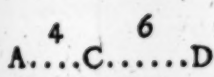


If from an even number AB, an odd number AC be taken away, the remaining number CB ſhall be odd.

For AC — 1 (AD) [a] is even; [b] therefore DB is even; [c] and conſequently CB (DB — 1) is odd. Which was to be demonſtrated.

[a] 7. *def.* 7.
[b] 24. 9.
[c] 7. *def.* 7.

PROP. XXVI.

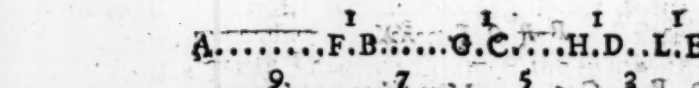


If from an odd number AB be taken away an odd number CB, that which remaineth AC ſhall be even.

For AB — 1 (AD) and CB — 1 (CD) [a] are even; [b] therefore AD — CD (AC) is even. Which was to be demonſtrated.

[a] 7. *def.* 7.
[b] 24. 9.

PROP. XXII.

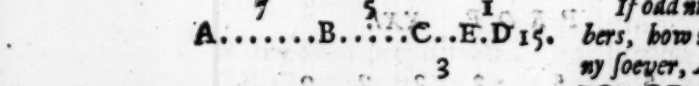


If odd numbers, how many soever, AB, BC, CD, DE, be added together, and their multitudes even, the whole also AE shall be even.

A unite being taken from each odd number, there will [a] remain AF, BG, CH, DI, even numbers, [b] and thence the number compounded of them will be even: add to them the [c] even number made of the remaining unites, and the [d] whole AE will thereby be even. Which was to be demonstrated.

[a] 7. def. 7.
[b] 21. 9.
[c] hyp.
[d] 21. 9.

PROP. XXIII.



If odd numbers, how many soever, AB, BC, CD, be added together, and the multitude of them be odd, the whole AD shall be odd.

For CD one of the odd numbers being taken away, the number compounded of the others AC [a] is even. Whereto add CD — 1, [b] the whole AE is also even; wherefore the unite being restored, the whole AD [c] will be odd. Which was to be demonstrated.

[a] 22. 9.
[b] 21. 3.
[c] 7. def. 7.

PROP.

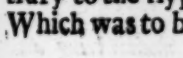
PROP. XXIV.

If an even number AB be taken away from an even number AC, that which remains BC ſhall be even.

For if BD (BC — 1) be odd, [a] BC (BD + 1) will be even. Which was to be demonſtrated. But if you ſay BD is even, becauſe AB [b] is even, [c] thence AD will be ſo; [a] and conſequently AC (AD + 1) will be odd, contrary to the Hypotheſis; therefore BC is even. Which was to be demonſtrated.

[a] 7. *def.* 7.
[b] *hyp.*
[c] 21. 9.

PROP. XXV.

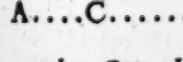


If from an even number AB, an odd number AC be taken away, the remaining number CB ſhall be odd.

For AC — 1 (AD) [a] is even; [b] therefore DB is even; [c] and conſequently CB (DB — 1) is odd. Which was to be demonſtrated.

[a] 7. *def.* 7.
[b] 24. 9.
[c] 7. *def.* 7.

PROP. XXVI.



If from an odd number AB be taken away an odd number CB, that which remaineth AC ſhall be even.

For AB — 1 (AD) and CB — 1 (CD) [a] are even; [b] therefore AD — CD (AC) is even. Which was to be demonſtrated.

[a] 7. *def.* 7.
[b] 24. 9.

PROP. XXVII.

If from an odd number AB be taken away an even number CB, the reſidue AC ſhall be odd.

For AB — 1 (DB) [a] is even, and CB is ſuppoſed to be even; [b] therefore the reſidue CD is even: [c] therefore CD + 1 (CA) is odd. Which was to be demonſtrated.

[a] 7. *def.* 7.
[b] 24. 9.
[c] 7. *def.* 7.

PROP. XXVIII.

A, 3.
B, 4.
AB, 12.

If an odd number A multiplying an even number B produce a number AB, the number produced AB ſhall be even.

For AB [a] is compounded of the odd number A taken as many times as a unite is contained in B an even number. [b] Therefore AB is an even number.

[a] *hyp. and* 15 *def.* 7.
[b] 21. 9.

Scholium.

In like manner, if A be an even number, AB ſhall be an even number alſo.

PROP. XXIX.

A, 3.
B, 5.
AB, 15.

If an odd number A multiplying an odd number B, produce a number AB, the number produced AB ſhall be odd.

For

For AB [a] is compounded of the odd number B taken as often as a unite is included in A, likewise an odd number. [b] Therefore AB is an odd number. Which was to be demonſtrated.

[a] 15. def. 7.
[b] 23. 9.

Scholium.

B, 12. (C, 4.
A, 3.

1. *An odd number A meaſuring an even number B, meaſures the ſame by an even number C.*

For if C be affirmed to be odd, then becauſe [a] B = AC; [b] therefore B ſhall be odd, againſt the Hypotheſis.

[a] 9. ax. 7.
[b] 29. 9.

B, 15. (C, 5.
A, 3.

2. *An odd number A meaſuring an odd number B, meaſures the ſame by an odd number C.*

For if C be ſaid to be even, [a] then AC, or B will be even, contrary to the Hypotheſis.

[a] 28. 9.

B, 15. (C, 5.
A, 3.

3. *Every number (A and C) that meaſures an odd number B, is it ſelf an odd number.*

For if either A or C be affirmed to be even, B [a] ſhall be an even number, againſt the Hypotheſis.

[a] 28. 9.

PROP. XXX.

B, 24. (C, 8. D, 12. (E, 4.
A, 3. A, 3.

If an odd number A meaſure an even number B, it ſhall alſo meaſure the half of it D.

[a] Let $\frac{B}{A}$ be $=C$; [b] then C is an even number. Therefore let E be $=\frac{1}{2}C$, then B [c] $=$ CA [d] $= 2EA$ [e] $= 2D$: [f] therefore $EA = D$; [g] and consequently $\frac{D}{A} = E$. Which was to be demonſtrated.

[a] *hyp.*
[b] 1. *ſchol.* 29. 9.
[c] 9. *ax.* 7.
[d] 1. 2.
[e] *hyp.*
[f] 7. *ax.* 1.
[g] 7. *ax.* 7.

PROP. XXXI.

A, 5. B, 8. C, 16. D---

If an odd number A be prime to any number B, it ſhall alſo be prime to the double thereof C.

If it be poſſible, let ſome number D meaſure A and C, [a] then D meaſuring the odd number A ſhall be odd it ſelf, [b] and ſo ſhall meaſure B the half of the even number C; therefore A and B are not prime one to another. Which is againſt the Hypotheſis.

[a] 3. *ſchol.* 29. 9.
[b] 30. 9.

Corollarium.

It follows from hence that an odd number which is prime to any number of double progreſſion, is alſo prime to all the numbers of that progreſſion.

PROP. XXXII.

1. A, 2. B, 4. C, 8. D, 16.

All numbers A, B, C, D, &c. in double progreſſion from the binary are evenly even only.

It is evident that all theſe numbers 1, A, B, C, D, [a] are even, and [b] $\frac{\cdot:\cdot}{\cdot:\cdot}$, namely, in a

[a] 6. *def.* 7.
[b] 20. *def.* 7.

a double proportion; [c] and ſo every leſs meaſures the greater by ſome one of them. [d] Wherefore all are evenly even. But for that A is a prime number, [e] no number beſide theſe ſhall meaſure any of them. Therefore they are evenly even only. Which was to be demonſtrated.

[c] 12. 9.
[d] 8. *def.* 7.
[e] 13. 9.

PROP. XXXIII.

A, 30. B, 15. D---E--

If of a number A, the half B be odd, the ſame A is evenly odd only.

Being an odd number B [a] meaſures A by two an even number; [b] therefore B is evenly odd. If you affirm it to be evenly even, [c] then ſome even number D meaſures it by an even number E; whence 2 B [d] = A [d] = DE: [e] wherefore 2. E :: D. B; and therefore as 2 [f] meaſures the even number E, [g] ſo D an even number meaſures B an odd. Which is impoſſible.

[a] *hyp.*
[b] 9. *def.* 7.
[c] 8. *def.* 7.
[d] 9. *ax.* 7.
[e] 9. 7.
[f] 6. *def.* 7.
[g] 20. *def.* 7.

PROP. XXXIV.

A, 24.

If an even number A be neither doubled from two, nor have its half part odd, it is both evenly even and evenly odd.

It is undoubtable, that A is evenly even, becauſe the half of it is not odd. But becauſe, if A be divided into two equal parts, and ſo continuing the bipartition, we ſhall at length light upon ſome [a] odd number (not upon the number two, becauſe A is not ſuppoſed to be doubled upward from two) which ſhall meaſure A by an even number; (for [b] otherwiſe A it ſelf ſhould be odd, againſt the Hypotheſis.) Therefore A is alſo evenly odd. Which was to be demonſtrated.

[a] 7. *def.* 7.
[b] 1. *ſch.* 29. 9.

PROP. XXXV.

A.....8

4 8

B....F.....G 12.

C.....18.

9 6 4 8

D.....H.....L....K.....N 27.

If there be numbers in continual proportion how many soever, A, BG, C, DN, and the number FG be taken from the second, and KN from the last, equal to the first A; as the excess of the second BF is to the first A, so shall the excess of the last DK be to all the numbers that precede it, A, BG, C.

From DN take NL = BG, and NH = C.

[a] *hyp.* Because DN . C (HN) [a] :: HN . BG (LN)

[b] 17. 5. [a] :: LN (BG) . A (KN;) [b] therefore by dividing each, shall DH . HN :: HL . LN :: LK.

[c] 12. 5. KN; [c] wherefore DK . C + BG + A :: LK

[d] 3. *ax.* 1. ([d] BF) . KN (A.) Which was to be demonstrated.

Corollarium.

Hence [e] by compounding, DN + BG + C.

[e] 18. 5. A + BG + C :: BG . A.

PROP. XXXVI.

1. A, 2. B, 4. C, 8. D, 16.
E, 31. G, 62. H, 124. L, 248. F, 496.
M, 31. N, 465.
P---- Q---

If from a unite be taken how many numbers soever 1, A, B, C, D, in double proportion continually, until the whole added together E be a prime num-

number; and if this whole E *multiplying the laſt produce a number* F, *that which is produced* F *ſhall be a perfect number.*

Take as many numbers E, G, H, L, likewiſe in double proportion continually; then [a] of equality A.D :: E.L; [b] therefore AL = DE [c] = F, [d] whence $L = \frac{F}{2}$. Wherefore E, G, H, L, F, are ∴ in double proportion. Let G — E be = M, and F — E = N; [e] then M.E :: N.E + G + H + L. [f] But M = E; [g] therefore N = E + G + H + L: [h] therefore F = 1 + B + C + D + E + G + H + L = E + N. Moreover becauſe D [k] meaſures DE (F) [l] therefore every one, 1, A, B, C, [m] meaſuring D, as [m] alſo E, G, H, L, does meaſure F. And further, no other number meaſures the ſaid F. For if there do, let it be P, which meaſures F by Q; [n] therefore PQ = F = DE: [o] therefore E . Q :: P . D; therefore ſeeing A a prime number meaſures D, [p] and ſo no other P meaſures the ſame; [q] conſequently E does not meaſure Q. Wherefore E being ſuppoſed a prime number, [r] it ſhall be prime to Q; [s] wherefore E and Q are the leaſt in their proportion; [t] and ſo E meaſures P as many times as Q does D: [u] therefore Q is one of them A, B, C. Let it be B, ſeeing then of equality B.D :: E.H; [x] and ſo BH = DE = F = PQ: [x] and ſo alſo Q.B :: H.P; [y] therefore H = P: therefore P is alſo one of them A, B, C, &c. againſt the Hypotheſis. Wherefore no other beſide the foreſaid numbers meaſures F, and [z] conſequently F is a perfect number. Which was to be demonſtrated.

[a] 14. 7. [b] 19. 7. [c] *hyp.* [d] 7. *ax.* 7. [e] 35. 9. [f] 3. *ax.* 1. [g] 14. 5. [h] 2. *ax.* 1. [k] 7. *ax.* 7. [l] 11. *ax.* 7. [m] 11. 9. [n] 9. *ax.* 7. [o] 19. 7. [p] 13. 9. [q] 20. *def.* 7. [r] 21. 7. [s] 23. 7. [t] 21. 7. [u] 13. 7. [x] 19. 7. [y] 14. 5. [z] 22. *def.* 7.

The End of the Ninth Book.

THE TENTH BOOK OF Euclid's *Elements*.

Definitions.

I. COmmenſurable magnitudes are thoſe which are meaſured by one and the ſame meaſure.

The note of commenſurability is ⊓, *as A* ⊓ *B; that is, the line A of* 8 *foot is commenſurable to the line B of* 13 *foot; becauſe D a line of one foot meaſures both A and B. Alſo* $\sqrt{18} \sqcap \sqrt{50}$; *becauſe* $\sqrt{2}$ *meaſures both* $\sqrt{18}$ *and* $\sqrt{50}$. *For* $\sqrt{\frac{18}{2}} = \sqrt{9} = 3$, *and* $\frac{50}{2} = \sqrt{25} = 5$; *wherefore* $\sqrt{18}$. $\sqrt{50} :: 3.5$.

II. Incommenſurable magnitudes are ſuch, of which no common meaſure can be found.

Incommenſurability is denoted by this mark ⊓; *as* $\sqrt{6} \sqcap \sqrt{25}$ (5;) *that is,* $\sqrt{6}$ *is incommenſurable to the number* 5, *or to a magnitude deſigned by that number; becauſe there is no common meaſure of them, as ſhall appear hereafter.*

III. Right

III. Right lines are commenſurable in power, when the ſame ſpace does meaſure their ſquares.

The mark of this commenſurability is ⊐; *as* AB ⊐ CD. *i. e. the line* AB *of* 6 *foot is in power commenſurable to the line* CD, *which is expreſſed by* √ 20, *becauſe* E *the ſpace of one foot ſquare does as well meaſure* ABq (36) *as the rectangle* XY (20) *to which the ſquare of the line* CD (√ 20) *is equal. The ſame note* ⊐ *ſometimes ſignifies commenſurable in power only.*

IV. Lines incommenſurable in power are ſuch, to whoſe ſquares no ſpace can be found to be a common meaſure.

This incommenſurability is denoted thus; 5 ⊐ υ√ 8. *i. e. the numbers or lines* 5, *and* υ√ 8 *are incommenſurable in power, becauſe their ſquares* 25 *and* √ 8 *are incommenſurable.*

V. From which it is manifeſt, that to any right line given right lines infinite in multitude are both commenſurable and incommenſurable; ſome in length and power, others in power only. The right line given is called a Rational line.

The note of which is ῥ.

VI. And lines commenſurable to this line, whether in length and power, or in power only, are alſo called Rational, ῥ.

VII. But ſuch as are incommenſurable to it, are called Irrational,

And denoted thus ἀῥ.

VIII. Alſo the ſquare which is made of the ſaid given right line is called Rational, ῥυ.

IX. And likewiſe ſuch figures as are commenſurable to it, are Rational, ῥα.

X. But ſuch as are incommenſurable, Irrational, ἀῥα.

XI. And those right lines also, which contain them in power, are Irrational ῥ.

Scholium.

That the last seven definitions may be rendred more clear by an example, let there be a Circle ADBP, whose semidiameter is CB, inscribe therein the sides of the ordinate figures, a of a Hexagone BP, of a Triangle AP, of a Square BD, of a Pentagone FD. Therefore, if according to the 5. def. *the semidiameter CB be the Rational line given, expressed by the number* 2. *to which the other lines BP, AP, BD, FD, are to be compared, then BP* [a] = *BC* = 2; *wherefore BP is* ῥ □ *BC, according to the* 6 def. *Also AP* [b] = √ 12 (*for ABq* (16) — *BPq* (4) = 12) *therefore AB is* ῥ ⊟ *BC likewise according to the* 6 def. *and APq* (12) *is* ῥν *by the* 9 def. *Moreover BD* [b] = √ *DCq* + *BCq* = √ 8; *whence BD is* ῥ ⊟ *BC, and BDq* ῥν. *Lastly, FDq* = 10 — √ 20 (*as shall appear by the praxis to be delivered at the* 10. 13.) *shall be* ῥν, *according to the* 10 def. *and consequently FD* = √ : 10 — √ 29 *is* ῥ, *according to the* 11 def.

[a] cor. 15. 4.
[b] 47. 1.

A Postulate.

That any magnitude may be so often multiplied, till it exceed any magnitude whatsoever of the same kind.

Axioms.

1. A magnitude measuring how many magnitudes soever, does also measure that which is composed of them.

2. A

2. A magnitude measuring any magnitude whatsoever, does likewise measure every magnitude which that measures.

3. A magnitude measuring a whole magnitude and a part of it taken away, does also measure the residue.

PROP. I.

Two unequal magnitudes AB, C, being given, if from the greater AB there be taken away more than half (AH) and from the residue (HB) be again taken away more than half (HI) and this be done continually, there shall at length be left a certain magnitude IB, less than the less of the magnitudes first given C.

[a] Take C so often, till its multiplex DE do somewhat exceed AB, and there be DF = FG = GE = C. Take from AB more than half HA, and from the remainder HB more than half HI, and so continually, till the parts AH, HI, IB, be equal in multitude to the parts DF, FG, GE. Now it is plain, that FE, which is not less than ½ DE, is greater than HB, which is less than ½ AB ⌐ DE. And in like manner GE, which is not less than ½ FE, is greater than IB ⌐ ½ HB; therefore C or GE ⌐ IB. Which was to be demonstrated. [a] *post.* 10.

The same may also be demonstrated, if from AB the half AH be taken away, and again from the residue HB the half HI, and so forward.

PROP.

PROP. II.

Two unequal magnitudes being given (AB, CD) if the leſs AB be continually taken from the greater CD, by an interchangeable ſubtraction, and the reſidue do not meaſure the magnitude going before, then are the magnitudes given incommenſurable.

If it be poſſible, let ſome magnitude E be the common meaſure. Then becauſe AB taken from CD, as often as it can be, leaves a magnitude FD leſs than it ſelf, and FD taken
[a] 1. 10. from AB leaves GB, and ſo forward; [a] therefore at length ſome magnitude GB ⌐ E ſhall
[b] *hyp.* be left; therefore E [b] meaſuring AB, [c] and ſo
[c] 2. *ax.* 10. CF, [b] and the whole CD, [d] ſhall alſo meaſure
[d] 3. *ax.* 10. the reſidue FD, [c] conſequently alſo AG; [d] wherefore it ſhall likewiſe meaſure the remainder GB, leſs than it ſelf. Which is abſurd.

PROP. III.

Two commenſurable magnitudes being given, AB, CD, to find out their greateſt common meaſure FB.

Take AB from CD, and the reſidue ED from AB, and FB from ED, till FB meaſure
[a] 2. 10. ED (which will come to paſs at length, [a] becauſe by the Hypotheſis AB ⊓ CD) FB ſhall be the magnitude required.

[b] *conſtr.* For FB [b] meaſures ED, [c] and ſo alſo AF;
[c] 2. *ax.* 10. but it meaſures it ſelf too, [d] therefore like-
[d] 1. *ax.* 10. wiſe AB, [c] and conſequently CE, [d] and ſo the whole CD. Wherefore FB is the common meaſure of AB, CD. If you affirm G to be a common meaſure greater than that, then G

mea-

measuring AB and CD, [e] measures also CE and [f] the remainder ED, [e] and so AF; and [f] consequently the remainder FB, the greater the less. Which is absurd.

[e] 2. *ax.* 10.
[f] 3. *ax.* 10.

Corollarium.

Hence, A magnitude that measures two magnitudes, does also measure their greatest common measure.

PROP. IV.

Three commensurable magnitudes being given, A, B, C, to find out their greatest common measure.

[a] Find out D the greatest common measure of any two A, B; [a] also E the greatest common measure of D and C: therefore E is the magnitude sought for.

[a] 3. 10.

[a] For it is clear, E measuring D and C, [b] does measure the three, A, B, C. Conceive another magnitude F greater than that to measure them; [c] then F measures D, [c] and consequently E the greatest common measure of D, and C, the greater the less. Which is absurd.

[b] *constr.* & 2. *ax.* 10.
[c] *cor.* 3. 10.

Corollarium.

Hence also it appears, that if a magnitude measure three magnitudes, it shall likewise measure their greatest common measure.

PROP.

PROP. V.

Commenſurable magnitudes A, B, have ſuch proportion one to another, as number hath to number.

[a] 3. 10. [a] C being found the greateſt common meaſure of A, B; as often as C is contained in A and B, ſo often is 1 contained in the numbers [b] 27. def. 7. D and E; [b] therefore C. A :: 1. D; wherefore inverſely A. C :: D. 1. [b] but likewiſe C. B :: 1. E; [c] therefore of equality A. B :: D. E :: N. N. Which was to be demonſtrated.

PROP. VI.

If two magnitudes A, B, have ſuch proportion one to another, as number C hath to number D, thoſe magnitudes A, B ſhall be commenſurable.

[a] ſch. 10. 6. What part 1 is of the number C, [a] that let
[b] conſtr. E be of A. Therefore becauſe E. A [b] :: 1. C,
[c] hyp. and A. B [c] :: C. D; [d] therefore of equality
[d] 22. 5. ſhall E. B :: 1. D. Wherefore ſeeing 1 [e] meaſures the number D, [f] likewiſe E meaſures B,
[e] 5. ax. 7. but it [g] alſo meaſures A; [h] therefore A ⊐ B.
[f] 20. def. 7. Which was to be demonſtrated.
[g] conſtr.
[h] 1. def. 10.

PROP. VII.

Incommenſurable magnitudes A, B, have not that proportion one to another, which number hath to number.

[a] 6. 10. If you affirm A. B :: N. N; [a] then A ⊐ B. Againſt the Hypotheſis.

PROP.

PROP. VIII.

If two magnitudes A, B, have not that proportion one to another, which number hath to number, those magnitudes are incommensurable.

Conceive A ⊏⊐ B; [a] then A . B :: N . N. Contrary to the Hypothesis. [a] 5. 10.

PROP. IX.

The squares described of right lines commensurable in length, have that proportion one to another, that a square number hath to a square number. And squares, which have that proportion one to another, that a square number hath to a square number, shall also have their sides commensurable in length. But such squares as are made of right lines incommensurable in length, have not that proportion one to another, which a square number hath to a square number. And squares which have not such proportion one to another, as a square number hath to a square number, have not their sides commensurable in length.

1. *Hypothesis.* A ⊏⊐ B. I say, Aq . Bq :: Q . Q.

For [a] let A . B :: number E. number F; therefore $\frac{Aq}{Bq}$ ([b] $\frac{A}{B}$ twice) [c] $= \frac{E}{F}$ twice. [d] $= \frac{Eq}{Fq}$; [e] therefore Aq . Bq :: Eq . Fq :: Q . Q. Which was to be demonstrated.

[a] 5. 10.
[b] 20. 6.
[c] *sch.* 23. 5.
[d] 11. 8.
[e] 11. 5.

2. *Hypothesis.* Aq . Bq :: Eq . Fq :: Q . Q. I say, A ⊏⊐ B. For $\frac{A}{B}$ twice ([f] $\frac{Aq}{Bq}$) [g] $= \frac{Eq}{Fq}$ [h] $= \frac{E}{F}$ twice; [i] therefore A . B :: E . F :: N . N:

[f] 20. 6.
[g] *hyp.*
[h] 11. 8.
[i] *sch.* 23. 5.

[k] 6. 10. N.N; [k] wherefore A ⊐ B. Which was to be demonstrated.

3. *Hypothesis.* A ⊐ B: I deny that Aq . Bq :: Q.Q. For suppose Aq . Bq :: Q.Q; then A ⊐ B, as is shewn before, against the Hypothesis.

4. *Hypothesis.* Not Aq . Bq :: Q.Q. I say, that A ⊐ B. For conceive A ⊐ B; then Aq . Eq :: Q.Q, as above, against the Hypothesis.

Corollarium.

Lines ⊐ are also ⊐; but not on the contrary. And lines ⊐ are not therefore ⊐; but ⊐ are also ⊐.

PROP. X.

If four magnitudes be proportional (C . A :: B . D) and the first C be commensurable to the second A, the third B shall be commensurable to the fourth D. And if the first C be incommensurable to the second A, also the third B shall be incommensurable to the fourth D.

[a] 5. 10. If C ⊐ A, [a] then C . A :: N . N [b] :: B . D;
[b] 6. 10. [b] therefore B ⊐ D. But if ⊐ A, [c] then
[c] 7. 10. shall not C . A :: N . N :: B . D; [d] wherefore
[d] 8. 10. B ⊐ D. Which was to be demonstrated.

Lemma 1.

To find out two plane numbers, not having the proportion which a square number hath to a square.

Any two plane numbers not like will satisfie this Lemma, as those numbers which have superparticular, superbipartient, or double

double proportion; or any two prime numbers. *See Schol.* 27. 8.

Lemma 2.

To find out a line HR, to which a right line given KM hath the proportion of two numbers given B, C.

[a] Divide KM into as many equal parts as there are unites in the number B; and let as many of these, as there are unites in the number C, [b] make the right line HR; it is manifest that KM. HR :: B. C.

[a] *Sch.* 10. 6.
[b] 3. 1.

Lemma 3.

To find out a line D, to the square of which the square of a right line given KM hath the proportion of two numbers given B, C.

Allow B. C [a] :: KM. HR; and between KM and HR, [b] find a mean proportional D. Therefore KMq. Dq [c] :: KM. HR [d] :: B. C.

[a] 2. *lem.* 10. 10.
[b] 13. 6.
[c] 20. 6.
[d] *constr.*

PROP. XI.

To find two right lines incommensurable to a right line given A, one D in length only, the other E in power also.

1. Take the numbers B, C, [a] so that there be not B . C :: Q . Q; [b] and let B . C :: Aq. Dq, [c] it is plain that A ⊓ D. But Aq [d] ⊓ Dq. Which was to be done.

[a] 1. *lem.* 10. 10.
[b] 3. *lem.* 10. 10.
[c] 9. 10.
[d] 6. 10

2. [d] Make A . E :: E . D. I say, Aq ⊓ Eq. For A. D [e] :: Aq. Eq; therefore since A ⊓ D, as before: [f] therefore Aq ⊓ Eq. Which was to be done.

[d] 13. 6.
[e] 20. 6.
[f] 10. 10.

PROP.

PROP. XII.

D, 18. E, 8. F, 2. G, 3. H, 5. I, 4. K, 6.

Magnitudes (A, B) commensurable to the same magnitude C, are also commensurable one to the other.

[a] 5. 10. Because A ⌑ C, and C ⌑ B, [c] let A . C :: N . N :: D . E, and C . B :: N . N :: F . G:
[b] 4. 8. [b] take three numbers H, I, K, the least ⁝ in the proportions of D to E, and F to G. Now
[c] constr. because A . C [c] :: D . E [c] :: H . I, and C . B [c] ::
[d] 22. 5. F . G :: I . K; [d] therefore of equality A . B ::
[e] 6. 10. H . K :: N . N; [e] therefore A ⌑ B. Which was to be demonstrated.

Scholium.

12. 10. & def. 6. Hence, every right line commensurable to a rational line is also it self rational. And all right lines rational are commensurable to one another, at least in power. Also, every space
def. 9. commensurable to a rational space is rational too: and all rational spaces are commensurable
def. 7. and 10. one to another. But magnitudes, whereof one is rational, the other irrational, are incommensurable amongst themselves.

PROP. XIII.

If there be two magnitudes A, B, and one of them A commensurable to a third C, but the other B incommensurable, those magnitudes A, B are incommensurable.

[a] hyp. Conceive B ⌑ A, then being C [a] ⌑ A;
[b] 12. 10. [b] therefore C ⌑ B, against the Hypothesis.

PROP.

PROP. XIV.

If there be two magnitudes commensurable A, B, and one of them A incommensurable to any other magnitude C, the other also B shall be incommensurable to the same C.

Imagine B ⊡ C. then for that A [a] ⊡ B; [a] *hyp.*
[b] therefore A ⊡ C, against the Hypothesis. [b] 12. 10.

PROP. XV.

If four right lines be proportional (A . B : : C . D) and the first A be in power more than the second B by the square of a right line commensurable to it self in length, then also the third C shall be more in power than the fourth D by the square of a right line commensurable to it self in length. But if the first A be more in power than the second B by the square of a right line incommensurable to it self in length, then shall the third C be more in power than the fourth D by the square of a right line incommensurable to it self in length.

For because A . B [a] : : C . D; [b] therefore Aq. Bq : : Cq . Dq: [c] therefore by division Aq — Bq . Bq : : Cq — Dq . Dq; [d] wherefore √ : Aq — Bq . B : : √ : Cq — Dq . D; [e] and so inversely B . √ : Aq — B : : D √ : Cq — Dq; [f] therefore of equality A . √ : Aq — Bq : : C . √ : Cq — Dq; consequently if A ⊡, or ⊡ √ Aq — Bq, [g] then likewise C ⊡, or ⊡ √ : Cq — Dq. Which was to be demonstrated.

[a] *hyp.*
[b] 22. 6.
[c] 17. 5.
[d] 22. 6.
[e] *cor.* 4. 5.
[f] 22. 5.
[g] 10. 10.

PROP.

PROP. XVI.

If two magnitudes commensurable AB, BC, be composed, the whole magnitude AC shall be commensurable to each of the parts AB, BC. And if the whole magnitude AC be commensurable to either of the parts B, or BC, those two magnitudes given at first AB, BC, shall be commensurable.

[a] 3. 10. [b] 1. ax. 10. [c] 1. def. 10.

1. *Hypothesis.* [a] Let D be the common measure of AB, BC; [b] so D measures AC: and therefore AC ⊓ AB, and BC. Which was to be demonstrated.

[d] 3. ax. 10.

2. *Hypothesis.* [a] Let D be the common measure of AC, AB; [d] therefore D measures AC — AB (BC) and consequently AB ⊓ BC. Which was to be demonstrated.

Corollarium.

Hence it follows, if a whole magnitude composed of two be commensurable to any one of them, the same shall be commensurable to the other also.

PROP. XVII.

If two incommensurable magnitudes AB, BC, be composed, the whole magnitude also AC shall be incommensurable to either of the two parts AB, BC. And if the whole magnitude AC be incommensurable to one of them AB, the magnitudes first given AB, BC, shall be incommensurable.

[a] 3. ax. 10. [b] 1. def. 10.

1. *Hypothesis.* If it can be, let D be the common measure of AC, AB; [a] therefore D measures AC — AB (BC) [b] and therefore also AB ⊓ BC, against the Hypothesis.

2. Hypo-

2. *Hypothesis.* Conceive AB ⊓ BC; [c] therefore AC ⊓ AB. Against the Hypothesis. [c] 16. 10.

corollarium.

Hence also, if one magnitude, composed of two, be incommensurable to any one of them, the same also shall be incommensurable to the other.

PROP. XVIII.

If there be two unequal right lines AB, GK, and upon the greater AB a parallelogram ADB equal to the fourth part of a square made of the less line GK, and wanting in figure by a square, be applyed, and divide the said AB into parts commensurable in length AD, DB; then shall the greater line AB be more in power than the less GK, by the square of a right line FD commensurable in length to the greater. And if the greater AB be more in power more than the less GK by the square of the right line FD commensurable unto it self in length, and a parallelogram ADB equal to the fourth part of the square made of the less line GK, and wanting in figure by a square, be applyed to the greater AB, then shall it divide the same into parts AD, DB commensurable in length.

[a] Divide GK equally in H, and [b] make the rectangle ADB = GHq. Cut off AF = DB; then is ABq [c] = 4 ADB [d] (4 GHq or GKq) + FDq. Now in the first place, if AD ⊓ DB, then shall AB [e] ⊓ BD [e] ⊓ 2 DB [f] (AF + DB, or AB — FD) [g] therefore AB ⊓ FD. Which was to be demonstrated. But secondly, if AB ⊓ FD; [h] then shall AB ⊓ AB — FD (2 DB) [k] therefore AB ⊓ DB; [l] wherefore AD ⊓ DB. Which was to be demonstrated.

[a] 10. 1.
[b] 28. 6.
[c] 8. 2.
[d] *constr.* & 4. 2.
[e] 16. 10.
[g] *cor.* 16. 10
[h] *cor.* 16. 10
[k] 12. 10.
[l] 16. 10.

PROP.

PROP. XIX.

If there be two right lines unequal AB, GK, and to the greater AB be applyed a parallelogram ADB equal to the fourth part of a square made upon the leſs GK, and wanting in figure by a square, and alſo then applyed, divide the ſaid AB into parts AD, DB incommenſurable in length; the greater line AB ſhall be in power more than the leſs GK by the ſquare of the right line FD incommenſurable to the greater in length. And if the greater line AB be more in power than the leſs GK by the ſquare of a right line FD incommenſurable unto it ſelf in length, and if alſo upon the greater AB be applyed a parallelogram ADB equal to the fourth part of the ſquare of the leſs GK, and wanting in figure by a ſquare, then ſhall it divide the ſaid greater line AB into parts incommenſurable in length AD, DB.

Suppoſe all the ſame that was done and ſaid in the precedent Propoſition. Therefore firſt,

a 17. 10. If AD □ DB, a then ſhall AB □ DB. b Wherefore AB □ 2 DB (AB — FD) c therefore

b 13. 10. c *cor.* 17. 10. AB □ FD. Which was to be demonſtrated.

d 13. 30. e 17. 10. Secondly, If AB □ FD, then AB □ AB — FD (2 DB;) d wherefore AB □ DB, e and conſequently AD □ DB. Which was to be demonſtrated.

PROP. XX.

A rectangle BD comprehended under right lines BC, CD, rational and commenſurable in length, according to one of the foreſaid ways, is rational.

Let

Let A be given *ρ*, and [a] the ſquare BE described upon BC. Becauſe DC.CE (BC) [b] :: BD . BE, and DC [c] □ BC; [d] therefore ſhall the rectangle BD be □ ſquare BE; wherefore ſeeing the ſquare BE [e] □ Aq. ſhall alſo BD be □ Aq.; and ſo the rectangle BD is *ρ*. Which was to be demonſtrated.

[a] 46. 1.
[b] 1. 6.
[c] *hyp.*
[d] 10. 10.
[e] *hyp.* & 9.
def. 10.
[f] 12. 12.

Note, *There are three kinds of lines rational commenſurable one to another. For either, of two lines rational commenſurable in length one to the other, one is equal to the rational line propounded, or neither of them is equal to it; notwithſtanding both of them are commenſurable to it in length; or laſtly, both of them are commenſurable to the rational line given only in power. And theſe are the ways which the preſent theorem ſpeaks of.*

In numbers, let there be BC, $\sqrt{8}$ $(2\sqrt{2})$ and CD $\sqrt{:}8$ $(3\sqrt{2})$ then ſhall the rectangle BD $= \sqrt{144} = 12$.

PROP. XXI.

If a rational rectangle DB be applyed to a rational line DC, it makes the breadth thereof CB rational, and commenſurable in length to that line DC, whereto DB is applyed.

Let G be propounded *ρ*, and the ſquare DA deſcribed on BC, becauſe BD . DA :: [a] BC. CA; and BD, DA, [b] are *ρα*, [c] and ſo □; [d] therefore BC □ CA: but CD (CA) is *ρ*; [e] therefore BC is *ρ*. Which was to be demonſtrated.

[a] 1. 6.
[b] *hyp.*
[c] *ſch.* 12. 10.
[d] 10. 10.
[e] *ſch.* 12. 10.

In numbers, let there be the rectangle DB, 12. and DC, $\sqrt{8}$. then ſhall CB, $\sqrt{18}$. but $\sqrt{18} = 3\sqrt{2}$. and $\sqrt{8} = 2 \times \sqrt{2}$.

Lemma.

Lemma.

To find out two right lines rational commensurable only in power.

Let A be propounded ρ. [a] Take B ⊐ A, [a] and C ⊐ B; [b] it is clear that B and C are the lines required.

[a] 11. 10.
[b] *ſch.* 12. 10.

PROP. XXII.

A rectangle DB comprehended under right lines rational DC, CB commensurable in power only, is irrational: and the right line H, which containeth that rectangle in power is irrational, and called a Medial line.

Let G be the propounded ρ, and the ſquare DA deſcribed on DC, and let Hq = DB. Becauſe AC. CB [a] :: DA. DB, [b] and AC ⊓ CB, [c] ſhall be DA ⊓ DB (Hq.) [d] but Gq ⊓ DA; [e] therefore Hq ⊓ Gq: [f] wherefore H is ρ. Which was to be demonſtrated. And let it be called a Medial line, becauſe AC. H :: H. CB.

[a] 1. 6.
[b] *hyp.*
[c] 10. 10.
[d] *hyp.* & 9. *def.* 10.
[e] 13. 10.
[f] 11. 10.

In numbers, let there be DC, 3. and CB, √6. then ſhall the rectangle be DB (Hq) √54. wherefore H is ν√54.

The note of a medial line is μ, of a medial rectangle μγ, of more together μα.

Scholium.

Every rectangle that can be contained under two right lines rational commenſurable only in power, is medial, although it be contained under two right lines irrational: and every medial rectangle may be contained under two

two right lines rational, commenſurable only in power. As for example, the $\sqrt{24}$ is $\mu\nu$, becauſe it is contained under $\sqrt{3}$, and $\sqrt{8}$, which are ρ, $\square$; although it may be contained under $v\sqrt{6}$, and $v\sqrt{96}$ irrationals; for $\sqrt{24} = v\sqrt{576} = v\sqrt{6} \times v\sqrt{96}$.

PROP. XXIII.

If the rectangle BD made of a medial line A, be applyed on a rational line BC, it makes the breadth CD rational, and incommenſurable in length to the line BC, whereunto the rectangle BD is applyed.

Becauſe A is μ, [a] therefore ſhall Aq be equal to ſome rectangle (EG) contained under EF and FG ρ $\square$; [b] therefore BD = EG: [c] whence BC. EF :: FG. CD. [d] Therefore BCq. EFq :: FGq. CDq. But BCq and EFq [e] are $\rho\alpha$, [f] and ſo $\square$; [g] therefore FGq $\square$ CDq. Wherefore being FG is ρ, [h] therefore CD ſhall be ρ. Moreover, becauſe EF. FG [k] :: EFq. EG (BD;) for that EF $\square$ FG, [l] ſhall EFq be $\square$ BD. But EFq [m] $\square$ CDq; [n] therefore the rectangle BD $\square$ CDq. Whence being CDq. BD [o] :: CD. BC, [p] ſhall CD be $\square$ BC. Therefore, &c.

[a] *ſch.* 22. 10
[b] 1. *ax.* 1.
[c] 14. 6.
[d] 22. 6.
[e] *hyp.*
[f] *ſch.* 12. 10
[g] 10. 10.
[h] *ſch.* 12. 10
[k] 1. 6.
[l] 10. 10.
[m] *ſch.* 12. 10.
[n] 13. 10.
[o] 1. 6.
[p] 10. 10.

PROP. XXIV.

A right line B commenſurable to a medial line A is alſo a medial line.

Upon CD ρ [a] make the rectangle CE = Aq; [a] and the rectangle CF = Bq. Becauſe Aq (CE) is $\mu\nu$, [b] and CD ρ, [c] therefore ſhall the latitude DE be ρ $\square$ CD. But for that CE. CF [d] :: ED. DF, and CE [e] $\square$ CF; [f] therefore ED $\square$ DF: [g] therefore DF is ρ $\square$ CD.

[a] 11. 6.
[b] *hyp.*
[c] 23. 10.
[d] 1. 6.
[e] *hyp.*
[f] 10. 10.
[g] 12 & 13. 10.

h 21. 10. h Whence the rectangle CF (Bq) is $\mu\nu$, and ſo B is μ. Which was to be demonſtrated.

Obſ. *That the note* ⊐ *for the moſt part ſignifies commenſurable in power only, as in this and the precedent demonſtrations, &c.*

Corollarium.

Hereby it is manifeſt that a ſpace commenſurable to a medial ſpace, is alſo medial.

Lemma.

To find out two right lines medial A, B, commenſurable in length, and alſo two, A, C, commenſurable only in power.

a *lem.* 22. 10 *and* 13. 6. b 2. *lem.* 10 10. c 3. *lem.* 10. 20. d *conſtr. &* 24. 10. a Let A be any μ, b take B ⊐ A, and c C ⊐ A, d it appears to be done.

PROP. XXV.

A rectangle DB contained under DC, CB medial right lines commenſurable in length, is medial.

a 1. 6. b 10. 10. c 24. 10. Upon DC deſcribe the ſquare DA. Being AC. (DC). CB a :: DA. DB, and DC ⊐ CB; b ſhall DA ⊐ DB: c therefore DB is $\mu\nu$. Which was to be demonſtrated.

PROP. XXVI.

A rectangle AC comprehended under medial right lines AB, BC commenſurable only in power, is either rational or medial.

a 46. 1. b *cor.* 16. 6. Upon the lines AB, BC, a deſcribe the ſquares AD, CE; and upon FG ρ b make the rectangles FH = AD, b and IK = AC, b and LM = CE.

The

The ſquares AD, CE, that is, the rectangles FH, LM, [c] are μα and ▭; therefore GH, KM, having the ſame proportion [d] are ῥ, [e] and ▭; [f] therefore GH × KM is ῥν. But becauſe AD, AC, CE, that is, FH, IK, LM, [g] are ∴; [h] and ſo GH, HK, KM alſo ∴; [k] thence HKq = GH × KM: [l] therefore HK is ῥ, or ▭, or ▱IH (GF;) if ▭, [m] then the rectangle IK or AC is ῥν. But if ▱, [n] then AC is μν. Which was to be demonſtrated.

[c] *hyp.* & 24. 10.
[d] 23. 10.
[e] 10. 10.
[f] 20. 10.
[g] *ſch.* 22. 6.
[h] 1. 6.
[k] 17. 6.
[l] 12. 10.
[m] 20. 10.
[n] 22. 10.

Lemma.

If A and E be ▱ *only, Then firſt*, ſhall Aq, Eq, Aq + Eq, Aq — Eq [a] ▭. *And ſecondly*, Aq, Eq, Aq + Eq, Aq — Eq ▭ AE and 2 E. For A. E [b] :: Aq. AE [b] :: AE. Eq; therefore ſeeing A [c] ▭ E, [d] ſhall Aq ▭ AE, [e] and 2 AE; alſo Eq [d] ▭ AE, [e] and 2 AE. Wherefore becauſe Aq + Eq ▭ Aq and Eq; and Aq — Eq ▭ Aq and Eq. [f] Therefore ſhall Aq + Eq, [f] and Aq — Eq be ▭ AE, and 2 AE.

[a] *hyp. and* 16. 10.
[b] 1. 6.
[c] *hyp.*
[d] 10. 10.
[e] 14. 10.
[f] 14. 10.

Hence alſo thirdly, Aq, Eq, Aq + Eq, Aq — Eq, 2 AE [g] ▭ Aq + Eq + 2 AE; and Aq + Eq — 2 AE; [g] and Aq + Eq + 2 AE ▭ Aq + Eq — 2 AE. [h] (Q. A — E.)

[g] 14. 16. & 17. 10.
[h] *cor.* 7. 12.

PROP. XXVII.

A medial rectangle AB exceedeth not a medial rectangle AC by a rational rectangle DB.

Upon FF ῥ, [a] make EG = AB, [a] and EH = AC. The rectangles AB, AC, *i. e.* EG, EH, [b] are μα; [c] therefore FG and FH are ῥ ▱ EF. Whence, if KG, [d] *i. e.* DB be ῥν, [e] then ſhall HG be ▭ HK; [f] wherefore HG ▱ FH, [g] and conſequently FGq ▭ FHq. But FH is ῥ; [h] there-

[a] *cor.* 16. 6.
[b] *hyp.*
[c] 23. 10.
[d] 3. *ax.* 1.
[e] 21. 10.
[f] 13. 10.
[g] *lem.* 26. 1.

h *ſch.* 12. 10. h therefore is FG $\dot{\rho}$, but FG was $\dot{\rho}$ before. Which is contradictory.

Scholia.

1. *A rational rectangle AE exceeds a rational rectangle AD by a rational rectangle CE.*

a *hyp.* For AE a $\sqsupset$ AD, b therefore AE $\sqsupset$ CE;
b *cor.* 16. 10 c wherefore CE is $\dot{\rho}v$. Which, &c.
c *ſch.* 12. 10.

2. *A rational rectangle AD joyned with a rational rectangle CF makes a rational rectangle AF.*

a *ſch.* 12. 10 For AD a $\sqsupset$ CF; b wherefore AF $\sqsupset$ AD
b 16. 11. and CF; c and ſo AF is $\dot{\rho}v$. Which was to be
c *ſch.* 12. 10. demonſtrated.

PROP. XXVIII.

To find out medial lines (C and D) which contain a rational rectangle CD.

a *lem* 21. 10 a Take A and B $\dot{\rho}$ $\sqcap$, b make A . C :: C . B,
b 13. 6. c and A . B :: C . D. I ſay, the thing re-
c 12. 6. quired is done. For AB (Cq) d is μv, d whence
d 22. 10. C is μ, but being that A . B c :: C . D; f there-
e *conſtr.* fore C $\sqcap$ D, g and conſequently D is μ. More-
f 10. 10. over by inverſion A . C :: B . D. *i. e.* C . B :: B . D;
g 24. 10. h therefore Bq = CD. But Bq is $\dot{\rho}v$, h there-
h 17. 6. fore CD is $\dot{\rho}v$. Which was to be done.
h *ſch.* 12. 10.

In numbers, let A be $\sqrt{2}$, and B $\sqrt{6}$; therefore C is $v\sqrt{12}$. make $\sqrt{2} . \sqrt{6} :: v\sqrt{12} . D$. or $v\sqrt{4} . v\sqrt{36} :: v\sqrt{12} . D$; then ſhall D be $v\sqrt{108}$. But $v\sqrt{12} \times v\sqrt{108} = v\sqrt{1296} = \sqrt{36} = 6$; therefore CD is 6; likewiſe C . D :: 1 . $\sqrt{3}$. wherefore C $\sqcap$ D.

PROP.

PROP. XXIX.

To find out medial right lines commensurable in power only, D and E, containing a medial rectangle DE.

[a] Take A, B, C, ρ ☐. make A. D [b] :: D. B, [c] and B. C :: D. E. I say, the thing desired is performed.

For AB [d] = Dq, and AB [e] is μη; therefore D is μ, and B [f] ☐ C, [g] whence D ☐ E; therefore [h] E is μ. Moreover B . C [f] :: D. E, and by inversion B. D :: C. E. *i.e.* D. A :: C. E; [l] therefore DE = AC. But AC [m] is μη; therefore DE is μη. Which was to be done.

In numbers, let A be 20, and B, √200, and C, √80. Therefore D is √√80000; and E υ√12800. Therefore DE = √√1024000000 = 32000, and D . E :: √10 . 2. wherefore D ☐ E.

[a] *lem.* 21. 10
[b] 13. 6.
[c] 12. 6.
[d] 17. 6.
[e] 22. 20.
[f] *constr.*
[g] 10. 10.
[h] 24. 10.
[k] *constr.* & *cor.* 4. 5.
[l] 16. 6.
[m] 22. 6.

Scholium.

A, 6.	C, 12.
B, 4.	D, 8.
AB, 24.	CD, 96.

A, 6.	C, 5.
B, 4.	D, 8.
AB, 24.	CD, 40.

To find out two plane numbers, like or unlike. Take any four numbers proportional A. B :: C. D, it is manifest that AB and CD are like plane numbers. And you may find out as many unlike plane numbers, as you please, by help of *Schol.* 27. 8.

Lemma.

1. *To find out two ſquare numbers (DEq and CDq) ſo that the number compoſed of them (CEq) be ſquare alſo.*

Take AD, DB like plane numbers (of which let both be equal, or both odd) *viz.* AD, 24. and DB, 6. The total of theſe (AB) is 30; the difference (FD) 18, half of which (CD) is 9. [a] Now the like plane numbers AD, DB, have one mean number proportional, namely, DE; therefore it is evident that every of thoſe numbers CE, CD, DE, are rational, and by conſequence CEq ([b] CDq + DEq) is the ſquare number required.

[a] 18. 8.

[b] 47. 1.

Whereby it will be eaſie to find out two ſquare numbers, the exceſs of which is a ſquare or not a ſquare number; namely, by the ſame conſtruction [c] ſhall CEq — CDq be = DEq.

[c] 3. ax. 1.

But if AD, DB be plane numbers unlike, the medial proportional line (DE) ſhall not be a rational number, and ſo neither ſhall the exceſs (DEq) of the ſquare numbers, CEq, CDq, be a ſquare number.

Lemma 2.

2. *To find out two ſuch ſquare numbers B, C, as the number compounded of them D is not ſquare. Alſo to divide a ſquare number A into two numbers B, C, not ſquares.*

A, 3. B, 9. C, 36. D, 45.

1. Take any ſquare number B, and let C be = 4B, and D = B + C. I ſay the matter is done.

For

For B is Q. by the conſtruction: Likewiſe becauſe B.C::1.4::Q.Q. [a] therefore C alſo ſhall be a ſquare number. But becauſe B+C. (D) C::5.4:: not Q. Q. [b] Therefore ſhall not D be a ſquare number. Which was to be done.

[a] 24. 8.

[b] *cor.* 24. 8.

A, 36. B, 24. C, 12. D, 3. E, 2. F, 1.

2. Let A be ſome ſquare number. Take D, E, F, plane numbers unlike, and let D be =E+F. make D.E::A.B, and D.F::A.C. I ſay the thing required is done.

For becauſe D.E+F::A.B+C, and D=E+F; [a] therefore ſhall A=B+C. Now ſuppoſe B to be ſquare, [b] then A and B, [c] and conſequently D and E are like plane numbers. Which is contrary to the Hypotheſis.

[a] 14. 5.

[b] 21. *def.* 7.

[c] 26. 8.

The ſame abſurdity will follow if C be ſuppoſed a ſquare number. Therefore, &c.

PROP. XXX.

To find out two ſuch rational right lines AB, AF, commenſurable only in power, as the greater AB ſhall be in power more than the leſs AF by the ſquare of a right line BF commenſurable in length to the greater.

Let AB be the line given ῥ. [a] Take the ſquare numbers CD, CE, ſo that CD—CE (ED) be not Q; [b] and let there be CD.ED::ABq. AFq. In a circle deſcribed upon the diameter AB [c] draw AF, and alſo BF. Then I ſay AB, AF, are the lines required.

[a] 1. *lem.* 29. 10.

[b] 3. *lem.* 10. 10.

[c] 1. 4.

For ABq. AFq [d] :: CD. ED; [e] therefore ABq ⊟ AFq. but AB is ῥ, [f] therefore AF is alſo ῥ. But becauſe CD is Q: and ED not Q: [g] there-

[d] *conſtr.*

[e] 6. 10.

[f] *ſch.* 12

[g] 9. 10. [g] therefore ſhall AB be ⊏ AF. Moreover by
[h] 31. 3. reaſon of the [h] right angle AFB, is ABq [k] =
[k] 47. 1. AFq + BFq; therefore ſeeing ABq. AFq :: CD. ED; by converſion of proportion ſhall ABq.
[l] 9. 10. BFq :: CD. CE :: Q. Q : [l] therefore AB ⊐ BF. Which was to be done.

In numbers, let there be AB, 6; CD, 9; CE, 4; wherefore ED, 5. Make 9 . 5 :: 36. (Q: 6) AFq; then AFq ſhall be 20. and conſequently AF √ 20. Therefore BFq = 36 — 20 = 16; wherefore BF is 4.

PROP. XXXI.

To find out two rational lines AB, AF commenſurable only in power, ſo that the greater AB ſhall be in power more than the leſs AF by the ſquare of a right line BF incommenſurable in length to the greater.

[a] 2. *lem.* 29. 10. Let AB be the line given ρ. [a] Take the ſquare numbers CE, ED, ſo that CD = CE + ED be not Q. And in the reſt follow the conſtruction of the preceding Propoſition. I ſay then the thing required is done.

For, as above, AB, AF, are [c] ρ ⊐; alſo ABq . BFq :: CD . ED; therefore being CD
[b] 9. 10. is not Q. AB, BF [b] ſhall be ⊐. Which was to be done.

In numbers, let there be AB, 5. CD, 45. CE = 36. ED = 9. Make 45 . 9 :: 25 . (ABq.) 5 (AFq.) therefore AF = √ 5. conſequently BFq = 45 — 25 = 20; wherefore BF = √ 20.

PROP.

PROP. XXXII.

To find out two medial lines C, D, commensurable only in power, comprehending a rational rectangle CD, so that the greater C be more in power than the lesser D by the square of a right line commensurable in length to the greater.

[a] Take A and B [ρ] ⊐; so as √Aq — Bq ⊏ A, [b] and make A . C :: C . B, [c] and A . B :: C . D. I say the thing is done.

For because A and [d] B are [ρ] ⊐, [e] therefore shall C ([f] √AB) be μ, [g] and thence also C ⊐ D; [h] therefore D is likewise μ. Furthermore, whereas A . B [d] :: C . D; and inversely A . C :: B . D :: C . B; and Bq is [ρν]: therefore shall CD ([k] Bq) be [ρν]. Lastly, because √Aq — Bq [d] ⊏ A, [l] shall √Cq — Dq be ⊏ C. Therefore, &c. But if √Aq — Bq ⊏ Aq, then shall √Cq — Dq be ⊏ C.

In numbers, let there be A, 8. B √48 (√: 64 — 16) therefore C = √AB = υ√ 3072. and D = υ√ 1728. wherefore CD = υ√ 5308416 = √2304.

a 30. 10.
b 13. 6.
c 12. 6.
d *constr.*
e 22. 10.
f 17. 6.
g 10. 10.
h 24. 10.
k 17. 6.
l 15. 10.

PROP. XXXIII.

To find out two medial lines D, E, commensurable in power only, comprehending a medial rectangle DE, so that the greater D shall be more in power than the less E, by the square of a right line commensurable to the greater in length.

[a] Take A and C [ρ] ⊐, so that √Aq — Cq ⊏ A. [b] Take also B ⊐ A and C, and make A . D [c] :: D . B [d] :: C . E; then D and E are the lines sought for.

a 30. 10.
b *lem.* 21. 10.
c 13. 6.
d 12. 6.

For because A and C [e] are ‘ρ, [e] and B ⊐ A and C; [f] therefore shall B be ‘ρ, and D ($\sqrt{AB}$) [g] shall be μ. But because A.D :: C.E; therefore inversely A.C :: D.E: wherefore seeing A ⊐ C, therefore D shall be ⊐ E; therefore E is μ. Furthermore, [l] being D.B :: C.E, and BC is μν; also DE, equal to it, is μν. Lastly, because A.C :: D.E. [e] seeing $\sqrt{Aq}$ — Cq ⊏ A; therefore $\sqrt{Dq}$ — Eq ⊏ D. Therefore, &c. But if $\sqrt{Aq}$ — Cq ⊏ A. then $\sqrt{Dq}$ — Eq ⊏ Eq.

[e] *constr.*
[f] *sch.* 12. 10.
[g] 22. 10.
[h] 10. 10.
[k] 24. 10.
[l] 22. 10.
[m] 16. 6.
[n] 15. 5.

In numbers, let there be A, 8, C $\sqrt{48}$. B $\sqrt{28}$. then D υ $\sqrt{3072}$. and E υ $\sqrt{588}$, wherefore D.E :: 2.$\sqrt{3}$. and DE = $\sqrt{1344}$.

PROP. XXXIV.

To find out two right lines AF, BF, incommensurable in power, whose squares added together make a rational figure, and the rectangle contained under them medial.

[a] Let there be found AB, CD, ‘ρ ⊐; so that $\sqrt{ABq}$ — CDq ⊏ AB. divide CD equally in G, [c] make the rectangle AEB = GCq. Upon AB the diameter draw the semicircle AFB, erect the perpendicular EF, and draw AF, BF. These are the lines required.

[a] 31. 10.
[b] 10. 1.
[c] 28. 6.
[d] 12. 6.
[e] *cor.* 8. 6. & 17. 6.

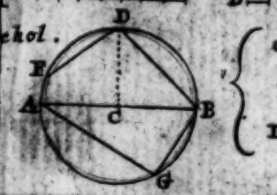
For AE . BE [d] :: BA × AE . AB × BE. But BA × AE [e] = AFq, and AB × BE = FBq; [f] therefore AE . EB :: AFq . FBq: therefore being AE [g] ⊏ EB, [h] AFq shall be ⊏ FBq. Moreover, ABq ([k] AFq + FBq) [l] is ‘ρν. Lastly, EFq [l] = AEB [l] = CGq; [m] therefore EF = CG: therefore CD × AB = 2 EF × AB. But CD × AB [n] is μν; [o] therefore AB × EF, [p] or AF × FB, is μν. Which was to be demonstrated.

[f] 7. 5.
[g] 19. 10.
[h] 10. 10.
[k] 31. 3. & 47. 1.
[l] *constr.*
[m] 1. *ax.* 1.
[n] 22. 10.
[o] 24. 10.
[p] *sch.* 22. 6.

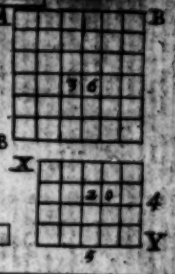
The

The Tenth Book of Euclid.

Def. I



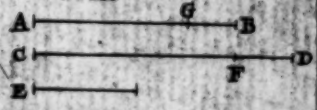
Def. III



Schol.

Prop. I

II



III

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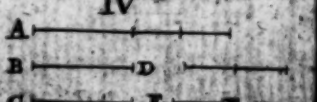

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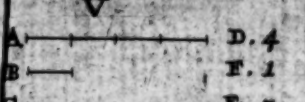

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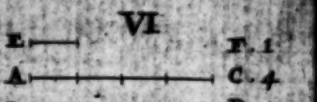
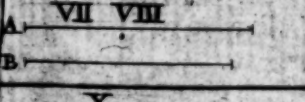


VII VIII



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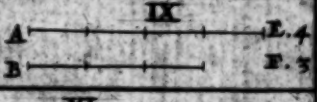

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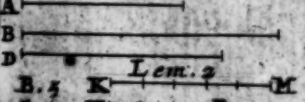

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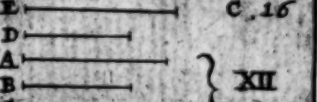
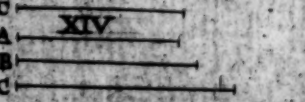
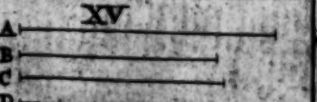


XIII

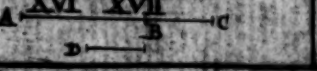
XIV



XV



XVI XVII



between 226 and 227.

XVIII. XIX
XX
XXI
Lem.
XXII
XXIII
XXIV
Lem.
XXV
XXVI
Lem.
XXVII
Schol. I
XXVII
Schol. II
XXVIII
XXIX
Lem.
Tho: Croft Junior Sculp.

The Explication of the same by numbers.

Let AB be 6. CD $\sqrt{12}$. then CG $= \sqrt{\frac{14}{4}} = \sqrt{3}$. But AE $= 3 + \sqrt{6}$. and EB $= 3 - \sqrt{6}$. whence AF shall be $\sqrt{:18 + 216}$. and FB $\sqrt{:18 - \sqrt{216}}$. Also AFq + FBq is 36, and AF × FB $= \sqrt{108}$.

But AE is found in this manner. Because BA (6). AF :: AF.AE; therefore 6 AE = AFq = AEq + 3 (EFq.) therefore 6 AE — AEq = 3. Put 3 + e = AE. then 18 + 6e — 9 — 6e — ee, that is 9 — ee = 3. or ee = 6. wherefore e $= \sqrt{6}$. and so AE $= 3 + \sqrt{6}$.

PROP. XXXV.

To find out two right lines AE, EB, incommensurable in power, whose squares added together make a medial figure, and the rectangle contained under them rational.

[a] Take AB and CF μ ⊐, so that AB × CF be ʻρν, and $\sqrt{\text{AEq} - \text{CFq}}$ ⊏ AB: and let the rest be done as in the preceding Proposition, AE, EB are the lines required. [a] 32. 10.

For, as it is shewn there, AEq ⊏ EBq; also ABq (AEq + EBq) is $\mu\nu$. And lastly, AB × CF [b] is ʻρν; [c] therefore also AB × DE, that is, AE × EB, is ʻρν. Therefore, &c. [b] *constr.* [c] *sch.* 12. 10. [d] *sch.* 22. 6.

PROP. XXXVI.

To find out two right lines BA, AC, incommensurable in power, whose squares added together make a medial figure, and the rectangle also contained under them medial, and incommensurable to the figure composed of the squares.

[a] Take

[a] 33. 10. [a] Take EC and EF μ $\sqsupset$, ſo that BC × EF be $\mu\nu$, and $\surd$ BCq — EFq $\sqsupset$ BC, and ſo forward, as in the prec. BA, AC, ſhall be the lines ſought for.

For (as above) BAq $\sqsupset$ ACq; alſo BAq + ACq is $\mu\nu$, and BA × AC is $\mu\nu$. Laſtly, [b] *conſtr.* BC [b] $\sqsupset$ EF, and [c] ſo BC $\sqsupset$ EG; likewiſe BC. [c] 13. 10. EG [d] :: BCq. BC × EG (BC × AD, or BA × AC) [d] 1. 6. [e] therefore BCq (ABq + ACq) $\sqsupset$ BA × AC. [e] 14. 10. Therefore, &c.

Scholium.

To find out two medial lines incommenſurable both in length and power.

[a] 36. 10. [a] Take BC μ, and let BA × AC be $\mu\nu$, and [q] 33. 6. $\sqsupset$ BCq (BAq + ACq) [b] make BA. H :: H. [c] 17. 6. AC; then I ſay, BC and H are μ $\sqsupset$. For BC [d] 14. 10. is μ, [a] and BA × AC ([c] Hq) is $\mu\nu$, wherefore H is alſo μ. [d] Likewiſe BA × AC $\sqsupset$ BCq; therefore Hq $\sqsupset$ BCq. Therefore, &c.

Here begin the ſenaries of lines irrational by compoſition.

PROP. XXXVII.

If two rational lines AB, BC, commenſurable only in power, be added together, the whole line AC is irrational, and is called a binomial line, or of two names.

[a] *hyp.* For becauſe AB [a] $\sqsupset$ BC, thence [b] ſhall ACq [b] *lem.* 26. 10. be $\sqsupset$ ABq. But AB [a] is [a] ρ; [c] therefore AC is [c] 11. *def.* 10. ρ. Which was to be demonſtrated.

PROP.

PROP. XXXVIII.

If two medial lines AB, BC, only in power commensurable, be compounded, and contain a rational rectangle, the whole line AC is irrational, and called a first bimedial line.

For being that AB [a] □ BC, [b] shall ACq be □ AB × BC, ρμ; [c] therefore AC is ρ. Which was to be demonstrated.

[a] *hyp.*
[b] *lem.* 26. 10.
[c] 11. *def.* 10

Lemma.

A rectangle AC, contained under a rational line AB and an irrational line BC, is irrational.

For if the rectangle AC be affirmed ρν, [a] then being AB is ρ, [b] the breadth BC shall be also ρ. Against the Hypothesis.

[a] *hyp.*
[b] 21. 10.

PROP. XXXIX.

If two medial lines AB, BC, commensurable only in power, containing a medial rectangle, be compounded, the whole line AC shall be irrational, and is called a second bimedial line.

Upon the propounded line DE ρ [a] make the rectangle DF = ACq; [b] and DG = AEq + BCq.

Because ABq [c] □ BCq, [d] therefore ABq + BCq, *i. e.* DG, □ ABq: but ABq [e] is μν; [e] therefore DG is μν. But the rectangle ABC is taken ρν, [e] and consequently 2 ABC ([f] HF) is ρν; [g] therefore EG and GF are ρ. Being also that DG [h] □ HF, and DG . HF :: [k] EG . GF; [l] therefore EG □ GF: [m] therefore the whole EF is ρ. [n] Wherefore the rectangle DF is ρν; [o] therefore √DF, *i.e.* AC, is ρ. Which was to be demonstrated.

[a] *cor.* 16. 6.
[b] 47. 1. & 11. 6.
[c] *hyp.*
[d] 16. 10.
[e] 24. 10.
[f] 4. 2.
[g] 23. 10.
[h] *lem.* 26. 10
[k] 1. 6.
[l] 10. 10.
[m] 37. 10.
[n] *lem.* 38. 10
[o] 11. *def.* 10

PROP.

PROP. XL.

If two right lines AB, BC incommenſurable in power, be added together, making that which is compoſed of their ſquares rational, and the rectangle contained under them medial, the whole right line AC is irrational, and is called a Major line.

For whereas ABq + BCq [a] is ‘ρν, and [b] □ 2 ABC [c] μν; and ſo ACq ([d] ABq + BCq + 2 ABC) [e] □ ABq + BCq ‘ρν; [f] therefore ſhall AC be ῥ. Which was to be demonſtrated.

[a] *hyp.*
[b] *ſch.* 12. 10
[c] *hyp.* & 24. 10.
[d] 4. 2.
[e] 17. 10.
[f] 11. *def.* 10

PROP. XLI.

If two right lines AC, CB, incommenſurable in power, be added together, having that which is made of their ſquares added together medial, and the rectangle contained under them rational, the whole right line AB ſhall be irrational, and is called A line containing in power a rational and a medial rectangle.

For 2 rectangles ACB [a] ‘ρν, [b] □ ACq + CBq [c] μν; [d] therefore 2 ACB [d] □ ABq: wherefore [e] AB is ῥ. Which was to be demonſtrated.

[a] *hyp. and ſch.* 12. 10.
[b] *ſch.* 12. 10
[c] *hyp.*
[d] 17. 10.
[e] 11. *def.* 10

PROP. XLII.

If two right lines GH, HK, incommenſurable in power be added together, having both that which is compoſed of their ſquares medial, and the rectangle contained under them medial, and incommenſurable to that which is compoſed of their ſquares, the whole right line GK is irrational, and is called A line containing in power two medial figures.

Upon

Upon the propounded line EB ‘ρ make the rectangles AF = GKq, and CF = GHq + HKq. Being GHq + HKq (CF) [a] is μν, the breadth CB [b] ſhall be ρ. Alſo becauſe two rectangles GHK (AD) [a] is μν; therefore AC [b] ſhall be ρ. Moreover, becauſe the rectangle AD [a] ▭ CF, [d] and AD . CF :: AC . CB; [e] thence ſhall [e] AG be ▭ CB, [f] wherefore A is [g] ρ; therefore the rectangle AF. *i. e.* GKq is ρν; [h] and conſequently GK is ρ. Which was to be demonſtrated.

[a] *hyp.*
[b] 23. 10.
[c] 4. 2.
[d] 2. 6.
[e] 20. 10.
[f] 37. 10.
[g] *lem.* 38. 10.
[h] 21. *def.* 10

PROP. XLIII.

A line of two names, or binomial, AB, can at one point only D be divided into its names AD, DB.

If it be poſſible, let the binomial line AB be divided at the point E, into other names AE, EB. It is manifeſt that the line AB is in both caſes divided unequally, ſince AD ▭ DB, and AE ▭ EB.

Becauſe the rectangles ADB, AEB [a] are μα; [a] and each of ADq, DBq, AEq, EBq is ρα; [b] and ſo ADq + DBq [b] and AEq + EBq are alſo ρα: [c] therefore ADq + DBq — : AEq + EBq. *i. e.* 2 AEB — 2 ADB is ρν. [d] therefore AEB — ADB is ρν: therefore μν exceeds μν by ‘ρν. [e] Which is abſurd.

[a] 37. 10.
[b] *ſcb.* 27. 10
[c] *ſcb.* 5. 2.
[d] *ſcb.* 12. 10
[e] 27. 10.

PROP. XLIV.

A firſt bimedial line AB, is in one point only D divided into its names AD, DB.

Conceive AB to be divided into other names AE, EB; whereupon every one ADq, DBq, EBq, will be [a] μα: and the rectangles ADB, AEB,

[a] 38. 10.

b *sch.* 27. 10. c *sch.* 5. 2. d 27. 10.

AEB, and the doubles of them, 'ρα; b therefore 2 AEB — 2 ADB. c *i. e.* ADq + DBq — : AEq + EBq is 'ρ. Which is absurd.

PROP. XLV.

A second bimedial line AB, is divided into its names AC, CB, only at one point C.

Suppose there were other names AD, DB. Upon the propounded line EF 'ρ make the rectangles EG = ABq, and EH = ACq + CBq: as also EK = ADq + DBq.

a 39. 10. b 16. & 24. 10. c 23. 10. d 24. 10. e 4. 2. f *lem.* 26. 10. g 1. 6. h 10. 10. k 37. 10.

Because ACq, BCq a are μα ▭; b ACq + CBq (EH) shall be μν: c therefore the breadth FH is 'ρ, a Moreover, the rectangle ACB, d and so 2 ACB (e IG) is μν; c therefore HG is also 'ρ. And since EH is f ▭ IG, g and EH. IG :: FH. HG; h therefore FH, HG shall be ▭: k therefore FG is a binomial, whose names are FH, HG. By the same reason FG is binomial, and the names of it FK, KG: contrary to the 43 of this Book.

PROP. XLVI.

A Major line AB is at one point only D divided into its names AD, DB.

a 40. 10. b *sch.* 27. 10 c *sch.* 5. 2. d 27. 10.

Imagine other names AE, EB, whereupon the rectangles ADB, AEB, are a μα; a and as well ADq + DBq, as AEq + EBq are 'ρα: b therefore ADq + DBq — : AEq + EBq, c *i. e.* 2 AEB — 2 ADB is 'ρ. d Which is impossible.

PROP.

PROP. XLVII.

A line AB containing in power a rational and a medial figure is divided at one point only D into its names AD, DB.

Conceive other names AE, EB; then both AEq + EBq, and ADq + DBq are $\mu\alpha$: [a] and the rectangles AEB, ADB are $\rho\alpha$; [b] therefore 2 AEB — 2 ADB, [c] *i. e.* ADq + DBq — : AEq + EBq is $\rho\nu$. [d] Which is absurd.

[a] 41. 10.
[b] *Sch.* 27. 10
[c] *Sch.* 5. 2.
[d] 27. 10.

PROP. XLVIII.

A line AB containing in power two medial rectangles, is at one point only C divided into its names AC, CB.

If you would divide AB into other names AD, DB, draw upon the line propounded EF ρ the rectangles EG = ABq. and EH = ACq + CBq, and EK = ADq + DBq; then because ACq + CBq, namely, EH, [a] is $\mu\nu$, [b] the breadth FH shall be ρ. Also because 2 ACB, [c] that is, IG, is [a] $\mu\nu$, HG [b] shall be likewise ρ. Therefore, whereas EH [a] $\square$ IG: and EH. IG [d] :: FH. HG, thence FH [e] shall be $\square$ HG; [f] therefore FG is a binomial, and the names of it FH, HG. In like manner FK, KG shall be the names of it, against the 43 of this Book.

[a] 42. 10.
[b] 23. 10.
[c] 4. 2.
[d] 1. 6.
[e] 10. 10.
[f] 37. 10.

Second Definitions.

A Rational line being propounded, and the binomial divided into its names, the greateſt of whoſe names is more in power than the leſs by a ſquare of a right line commenſurable to the greater in length; then,

I. If

I. If the greater name be commenſurable in length to the rational line propounded, the whole line is called a firſt binomial line.

II. But if the leſſer name be commenſurable in length to the rational line propounded, the whole line is called a ſecond binomial.

III. If neither of the names be commenſurable in length to the rational line propounded, it is called a third binomial.

Furthermore, if the greater name be more in power than the leſs by the ſquare of a right line incommenſurable to the greater in length, then

IV. If the greater name be commenſurable to the propounded rational line in length, it is called a fourth binomial.

V. If the leſſer name be ſo, a fifth.

VI. If neither, a ſixth.

PROP. XLIX.

To find out a firſt binomial line, EG.

[a] *ſch.* 29. 10 [b] 2. *lem.* 10. 10. [c] 3. *lem.* 10. 10. [d] *conſtr.* [e] 6. *def.* 10. [f] 6. 10. [g] *ſch.* 12. 10 [h] 9. 10. [k] 9. 10. [l] 1. *def.* 48. 10.

[a] Take AB, AC, ſquare numbers, whoſe exceſs CB is not Q. let D be propounded ρ. [b] Take EF ⊓ D, and [c] make AB. CB :: EFq. FGq; then EG ſhall be a firſt binomial.

For EF [d] ⊓ D; [e] therefore EF is [f] ρ: [g] alſo EFq ⊓ FGq; [h] therefore FG is alſo [k] ρ. Likewiſe [d] becauſe EFq. FGq :: AB. CB :: Q. not Q; [l] therefore EF ⊓ FG. Laſtly, becauſe by converſion of proportion, EFq. EFq — FGq :: AB. AC :: Q. Q; thence EF [k] ſhall be ⊓ √ EFq — FGq: [l] therefore EG is a firſt binomial. Which was to be done.

In numbers thus; let there be D 8. EF 6. AB 9. CB 5; wherefore becauſe 9. 5 :: 36. 20; therefore FG is √ 20, and conſequently EG is 6 + √ 20.

PROP.

PROP. L.

To find out a ſecond binomial line, EG.

Take AB and AC ſquare numbers, the exceſs of which is CB, not Q. Let D be the line propounded ῥ; take FG □ D, and make CB. AB :: FGq. EFq; then EG will be the line deſired. *Prove it as the prec.*

For FG □ D; wherefore FG is ῥ. Alſo EFq □ FGq; therefore EF is ῥ. Likewiſe becauſe FGq. EFq :: CB. AB :: not Q. Q; thence FG is □ EF. Laſtly, ſeeing CB. AB :: FGq. EFq, and inverſely AB. CB :: EFq. FGq: therefore as in the foregoing Prop. EF □ √ EFq — FGq. [a] whereby EG is a ſecond binomial. Which was to be done. [a] 2. *def.* 48. 10.

In numbers, let there be D 8, FG 10, AB 9. CB 5; then EF is √ 180. Wherefore EG is 10 + √ 180.

PROP. LI.

To find out a third binomial line, DF.

[a] Take AB, AC, ſquare numbers, the exceſs of which CB is not Q: and let L be a number not Q next greater than CB, *viz.* by a unite or two. Let G be the line propounded ῥ. [a] *ſch.* 29. 10.

[b] Make L. AB :: Gq. DEq, [b] and AB. CB :: DEq. EFq; then DF ſhall be a third binomial. [b] 3. *lem.* 10. 10.

For becauſe DEq [c] □ Gq, [d] DE is ῥ; alſo Gq. DEq :: L. AB :: not Q. Q: [e] therefore G □ DE. Likewiſe being that DEq [c] □ EFq, [d] alſo EF is ῥ. Moreover, becauſe DEq. EFq :: AB. CB :: Q not Q, [f] is DE □ EF; and being that by conſtruction and of equality Gq. EFq :: L. CB :: not Q. Q. (for [g] L and CB are [c] *conſtr.* 6. 10. [d] *ſch.* 12. 10 [e] 6. 10. [f] 9. 10. [g] *ſch.* 27. 8.

[h] 9. 10. are not like plane numbers;) [h] therefore ſhall G be alſo □ EF. Laſtly, as in the prec. Prop. [k] 3. *def.* 48. 10. √ DEq — EFq □ DE: [k] therefore DF is a third binomial. Which was to be done.

In numbers, let there be AB, 9. CB, 5. L, 6. G, 8. then ſhall be DE √ 96, and EF √ $\frac{225}{6}$. Wherefore DF = √ 96 + √ $\frac{225}{6}$.

PROP. LII.

To find out a fourth binomial line DF.

[a] *ſch.* 29. 10 [a] Take any ſquare number AB, and divide it into AC, CB not ſquares. Let G be the [b] 2 *lem.* 10. 10. line propounded ῥ, [b] take DE □ G, [c] and make AB. CB :: DEq. EFq: then DF ſhall be [c] 3. *lem.* 10. 10. a fourth binomial.

For, as in 49 of this Book, DF may be ſhewn [d] 9. 10. to be a binomial: and alſo becauſe by conſtruction and converſion of proportion DEq. [e] 4. *def.* 48. 10. DEq — EFq :: AB . AC :: Q, not Q. [d] ſhall DE be □ √ DEq — EFq. [e] Therefore DF is a fourth binomial.

In numbers, let G be 8, DE, 6. then EF ſhall be √ 24. Therefore DF is 6 + √ 24.

PROP. LIII.

To find out a fifth binomial line DF.

Take any ſquare number AB, whoſe ſegments AC, CB are not Q. Let G be the line propounded ῥ: take EF □ G, and make CB. AB :: EFq. DEq. Then ſhall DF be a fifth binomial.

For DE ſhall be a binomial, as in the 50 of this Book. And becauſe by conſtruction and inverſion, DEq. EFq :: AB. CB; and ſo by converſion of proportion, DEq. DEq — EFq

EFq :: AB . AC :: Q. not Q : [a] therefore ſhall DE be ± √ DEq — EFq. [b] Therefore DF is a fifth binomial. Which was to be done. [a] 9. 10. [b] 5. *def.* 48. 10.

In numbers, let there be G, 7, EF, 8. then DE ſhall be √ 54. wherefore DF is 6 + √ 54.

PROP. LV.

To find out a ſixth binomial line.

Take AC, CB, prime numbers, ſo that AC + CB (AB) be not Q. Take alſo any number ſquare L. Let G be the line propounded; [a] and make L. AB :: GQ. DEq, and AB. CB :: DEq. EFq; then DF ſhall be a ſixth binomial. [a] 3. *lem.* 10.

For DF may be demonſtrated binomial, as in the 51 of this Book: and alſo by reaſon that DE and EF ± G. Laſtly, likewiſe becauſe by conſtruction and converſion of proportion DEq . DEq — EFq :: AB . AC :: not Q. Q. (For AB is prime to AC, [b] and ſo unlike to it) [c] therefore DE ± √ DEq — EFq; [d] therefore DF is a ſixth binomial. Which was required. [b] *ſch.* 27. 8. [c] 9. 10. [d] 6. *def.* 48. 10.

In numbers, let there be G, 6. DE √ 48. then EF ſhall be √ 28. Wherefore DF is √ 48 + √ 28.

Lemma.

Let AD be a rectangle, and the ſide thereof AC divided unequally in E; alſo let the leſſer portion EC be equally divided in F, upon the line AE [a] *make the rectangle AGE = EFq: and from the points G, E, F* [b] *draw GH, EI, FK, parallel to AB.* [c] *Let the ſquare LM be made equal to the* [a] 28. 6. [b] 31. 1. [c] 14. 2.

the rectangle AH, and upon OMP produced the [a] *ſquare MN = GI, and let the right lines LOS, LQT, NRS, NPT be produced.*

I ſay, 1. MS, MT, *are rectangles.* For by reaſon of the right angles of the ſquares OMQ, RMP, [a] ſhall QMR be a right line; [b] therefore RMO, QEP, are right angles: wherefore the parallelograms MS, MT are rectangles.

[a] *ſch.* 15. 1.
[b] 13. 2.

[c] 2. *ax.* 1. 2. *Hence it is plain that LS* [c] = *LT, and conſequently that LN is a ſquare.*

3. *The rectangles SM, MT, EK, FD are equal.* For becauſe the rectangle AGE [d] = EFq; [e] thence ſhall AE. EF :: EF. GE, [f] and ſo AH. EK :: EK. GI. that is by conſtruction LM. EK :: EK. MN; [g] but LM. SM :: SM. MN. Therefore EK [h] = SM [k] = FD [l] = MT.

[d] *hyp.*
[e] 17. 6.
[f] 1. 6.
[g] *ſch.* 22. 6.
[h] 9. 5.
[k] 36. 1.
[l] 43. 1.

[m] 2. *ax.* 1. 4. *Hence LN* [m] *AD.*

5. *Being that EC is equally divided in F, it is plain that EF, FC, EC are* □.

[n] 16. 10.
[o] 18. & 16. 10.
[p] 10. 10.
6. If AE ⊐ EC, and AE □ √ AEq — ECq; [o] then ſhall AG, GE, AE, be □; alſo, becauſe AG. GE :: AH. GI: [p] therefore ſhall AH, GI, *i. e.* LM, MN, be □. Likewiſe thereupon,

[q] 14. 10.
[r] 10. 10.
7. OM □ MP. For by the Hypotheſis AE □ EC; [q] therefore EC □ GE. [q] Wherefore EF □ GE: but EF. GE :: EK. GI. [r] therefore EK □ GI; that is, SM □ MN. but SM. MN :: OM. MP. [r] Therefore OM □ MP.

[s] 19. & 17. 10.
8. If AE be ſuppoſed □ √ AEq — ECq, it is apparent that AG, GE, AE, are □; whence LM □ MN: for AG. GE :: AH. GI :: LM. MN.

Theſe being well conſidered, we ſhall eaſily diſpatch the ſix following Propoſitions.

PROP.

PROP. LV.

If a ſpace AD be contained under a rational line AB, and a firſt binomial line AC (AE + EC) the right line OP which containeth that ſpace in power is irrational, and is called a binomial line.

All that being ſuppoſed which is deſcribed and demonſtrated in the next foregoing *Lemma*, it is manifeſt that the right line OP containeth in power the ſpace AD. [a] Likewiſe AG, GE, AE, are □; therefore ſeeing AE [b] is ῥ □ AB, [c] ſhall alſo AG and GE be ῥ □ AB; [d] therefore the rectangles AH, GI, that is, the ſquares LM, MN are ῥα: therefore OM, MP are ῥ [e] □. [f] and conſequently OP is a binomial. Which was to be demonſtrated.

In numbers, let there be AB 5. AC 4 + √ 12. wherefore the rectangle AD = 20 + √ 300 = to the ſquare LN; therefore OP is √ 15 + √ 5. namely, a ſixth binomial.

[a] *hyp. and lem.* 54. 10.
[b] *hyp.*
[c] *ſch.* 12. 10
[d] 20. 10.
[e] *lem.* 54. 10.
[f] 37. 10.

PROP. LVI.

If a ſpace AD be comprehended under a rational line AB, and a ſecond binomial AC (AE + EC) the right line OP, which containeth that ſpace AD in power, is irrational, and called a firſt bimedial line.

The foreſaid *Lemma* of the 54 of this Book being again ſuppoſed, then ſhall OP be = √ AD: [a] alſo AE, G, GE are □. Therefore ſince AE [b] is ῥ □ AB, likewiſe AG, GE [c] ſhall be ῥ □ AB: therefore the rectangles AH, GI, *i. e.* OMq, MPq. [d] are μα. [e] Moreover OM □ MP. Laſtly, EF □ EC, and EC [f] □ AB. [g] where-

[a] *hyp. and lem.* 54. 10.
[b] *hyp.*
[c] *ſch.* 12. 10
[d] *lem.* 54. 10
[e] *hyp.* 12. 10

[e] 20. 10. [h] 38. 10.

[e] wherefore EK, *i. e.* SM, or OMP, is [ρν]. [h] Consequently OP is a first bimedial. Which was to be demonstrated.

In numbers, let there be AB, 5; and AC, $\sqrt{48}+6$. then the rectangle AD $=\sqrt{1200}+30=$ OPq. Therefore OP is $\upsilon\sqrt{675}+\upsilon\sqrt{75}$. *viz.* a first bimedial.

See Scheme 57.

PROP. LVII.

If a space AD be contained under a rational line AB and a third binominal line AC (AE + EC) the right line OP which containeth in power the space AD, is irrational, and called a second bimedial line.

[a] *hyp. and* 22. 10. [b] 39. 10.

As above, OPq $=$ AD; also the rectangles AH, GI, that is, OMP, MPq are $\mu\alpha$. [a] Likewise EK or OMP is $\mu\nu$. [b] Therefore OP is a second bimedial.

In numbers, let there be AB 5. AC $\sqrt{32}+\sqrt{34}$: wherefore AD is $\sqrt{800}+\sqrt{600}=$ OPq. and so OP is $\upsilon\sqrt{450}+\upsilon\sqrt{50}$. that is, a second bimedial.

PROP. LVIII.

If a space AD be comprehended under a rational line B and a fourth binominal AC (AE + EC) the line OP containing the space AD in power, is that irrational line which is called a Major line.

[a] *lem.* 54. 10 [b] *hyp. and* 20. 10. [c] *hyp. and* 22. 10. [d] 40. 10.

For again, OMq [a] $\square$ MPq; and the rectangle AI, *i. e.* OMq + MPq [b] is [ρν]. [c] Also EK or OMP is $\mu\nu$; [d] therefore OP ($\sqrt{AD}$) is a *Major* line. Which was to be demonstrated.

In

In numbers; let there be AB 5. and AC 4 + √ 8. then the rectangle AD is 20 + √ 200. Wherefore OP is √: 20 + √ 200.

PROP. LIX.

If a ſpace AD be contained under a rational line AB, and a fifth binomial AC, the right line OP, which containeth the ſpace AD in power, is that irrational line, which is a line containing a rational and a medial rectangle in power.

Again OMP □ MPq. and the rectangle AI or OMq + MPq is μν. [a] Likewiſe the rectangle EK or OMP is ρν. [b] therefore OP (√ AD) contains in power ρν and μν. Which was to be demonſtrated.

[a] as in the two.
[b] 41. 10.

In numbers, let there be AB 5, and AC 2 + √ 8. then the rectangle AD = 10 + √ 200 = OPq. Wherefore OP is √: 10 + 200.

PROP. LX.

If a ſpace AD be contained under a rational line AB, and a ſixth binomial AC (AE + EC) the line OP containing the ſpace AD in power is irrational, which containeth in power two medial rectangles.

As often before, OMq □ MPq, and OMq + MPq is μν. and alſo the rectangle (EK) OMP is μν. [a] therefore OP = √ AD contains in power 2 μα. Which was to be demonſtrated.

[a] 42. 10.

In numbers, let there be AB 5. AC √ 12 + √ 8. therefore the rectangle AD or OPq is √ 300 + √ 200. and ſo OP is √: √ 300 + √ 200.

Lemma.

Let a right line AB be unequally divided in C, and let AC be the greater portion, and upon some line DE apply the rectangles DF = ABq, and DH = ACq, and IF = CBq, and let LG be divided equally in M, and also MN drawn parallel to GF.

[a] 4, 2. and 3. ax. 1. Play 1. The rectangle ACB is = LN or MF. [a] For 2 ACB = EF.

[b] 7. 2. 2. DL □ LG. for DK (ACq + CBq) [b] □ = LF (2 ACB) therefore being DK, LF are of equal altitude, [c] DL shall be □ LG.

[c] 1. 6. [d] 16. 10. 3. If AC = CB, [d] then shall the rectangle DK be = ACq and CBq.

[e] *lem.* 26. 10. [f] 10. 10. [g] 1. 6. [h] 17. 6. [k] *hyp.* [l] 10. 10. [m] 18. 10. 4. *Also* DL = LG. for ACq + CBq [e] = 2 ACB, i. e. DK = LF. but DK. LF [f] :: DL. LG [g] therefore DL = LG.

5. *Moreover* DL = √ DLq — LGq. For ACq . ACB [h] :: ACB . CBq that is, DH . LN :: LN . IK. [k] wherefore DI . LM :: EM . IL [l] therefore DI × IL = LMq. therefore seeing ACq [m] = CBq, that is, DH = IK, and [n] so DI = IL, [n] shall DL be = √ DLq — LGq. Which was to be demonstrated.

[n] 19. 10. 6. *But if* ACq *be put* = CBq. [n] *then shall* DL *be* = √ DLq — LGq.

This Lemma is preparatory to the 6 following Propositions.

PROP. LXI.

The square of a binomial line (AC + CB) applied unto a rational line DE, makes the latitude DG a first binomial line.

Those things being supposed, which are described and demonstrated in the next preceding

Lemma;

Lemma; because AC, CB, [a] are ῥ ▭, [b] the rectangle DK shall be ▭ ACq. [c] and so DK is ῥγ. [d] therefore DL ▭ DE ῥ. but the Rectangle ACB, and so 2 ACB (LF) [e] is μγ. [f] therefore the latitude LG is ῥ ▭ DE. [g] therefore also DL ▭ LG also DL ▭ √ DLq. —LGq. from whence [h] it follows that DG is a first binomial. Which was to be demonstrated.

[a] *hyp.*
[b] *le.* 60. 10
[c] *ſcb.* 12. 10
[d] 21. 10.
[e] 22. & 24. 10.
[f] 23. 10.
[g] 13. 10.
[h] *le.* 60. 10
[k] 1 *def.* 48. 10.

PROP. LXII.

The ſquare of a firſt bimedial line (AC + CB) being applied to a rational line DE, makes the latitude DG a ſecond binomial line. Fig. pr. 61.

The aforeſaid *Lemma* being again ſuppoſed; [a] the rectangle DK ▭ ACq. [a] therefore DK is μγ. [b] therefore the latitude DL is ῥ ▭ DE. But because the rectangle ACB, and ſo LF (2 ACB) [c] is ῥγ, [d] ſhall LG be ῥ ▭ DE. [e] therefore DL, LG are ▭. [f] alſo DL ▭ √ DLq. — LGq. [g] from whence it is clear that DG is a ſecond binomial. Which was to be demonſtrated.

[a] 24. 10.
[b] 23. 10.
[c] *hyp. and ſcb.* 22. 10.
[d] 21. 10.
[e] 13. 10.
[f] *lem.* 60. 10.
[g] 2 *def.* 48. 10.

PROP. LXIII.

The ſquare of a ſecond bimedial line (AC + CB) applied to a rational line DE makes the breadth DG a third binomial line. Fig. pr. 61.

As in the *prec.* DL is ῥ ▭ DE. Furthermore because the rectangle ACB, and ſo LF (2 ACB) [a] is μγ. [b] therefore ſhall LG be ῥ ▭ DE. [c] Moreover DL ▭ LG. and alſo DL ▭ √ DLq —LGq. [d] therefore DG is a third binomial. Which was to be demonſtrated.

[a] *hyp. and* 24. 10.
[b] 23. 10.
[c] *lem.* 60. 10.
[d] 3 *def.* 48. 10.

PROP. LXIV.

Fig. pr. 61.

The square of a Major line (AC + CB) applied to a rational line DE, makes the breadth DG a fourth binomial line.

Again ACq + CBq. *i. e.* DK [a] is ῥν. [b] therefore DL is ʻρ ⊏ DE. also ACB, and so LF (2ACB) [c] is μν. [d] therefore LG is ʻρ ⊏ DE. [e] and consequently DL ⊏ LG. Lastly, because AC ⊐ BC. [f] shall DL be ⊏ DLq—LGq. [g] whence DG is a fourth binomial. Which was to be demonstrated.

[a] *hyp. and sch.* 12. 10.
[b] 21. 10.
[c] *hyp. and* 24. 10.
[d] 23. 10.
[e] 13. 10.
[f] *lem.* 60. 10.
[g] 4 *def.* 48. 10.

PROP. LXV.

Fig. pr. 61.

The square of a line containing in power a rational and a medial rectangle (AC + CB) applied to a rational line DE makes the latitude DG a fifth binomial.

Again, DK is μν. [a] therefore DL is ῥ ⊏ DE. also LF is ῥν. [b] therefore LG is ʻρ ⊏ DE. [c] therefore DL ⊏ LG. [d] likewise DL ⊏ √ DLq —LGq. [e] and so by consequence DG is a fifth binomial. Which was to be demonstrated.

[a] 23. 10.
[b] 21. 10.
[c] 13. 10.
[d] *lem.* 60. 10.
[e] 5 *def.* 48. 10.

PROP. LXVI.

Fig. pr. 61.

The square of a line containing in power two medial rectangles (AC + CB) applied to a rational line DE, makes the latitude DG a sixth binomial line.

As before, DL and LG and LG are ʻρ ⊏ DE. But for that ACq + CBq (DK) [a] ⊏ ACB, [b] and so DK ⊏ LF (2 ACB) and also DK. LF [c] :: DL. LG. [d] therefore shall DL be ⊏ LG. [e] Lastly,

[a] *hyp.*
[b] 14. 10.
[c] 1. 6.
[d] 10. 10.
[e] *lem.* 60. 10.

Laſtly DL □ √ DLq—LGq. [f] by which it appears that DG is a ſixth binomial. [f] 6 *def.* 48. 10.

Lemma.

Let AB, DE, be □. *and make AB. DE :: AC. DF.*

I ſay, 1. AC □ DF. as appears by 10. 10. alſo CB □ FE. [a] becauſe AB. DE :: CB. FE. [a] 19. 5.

2. AC. CB :: DF. FE. For AC. DF :: AB. DE :: CB. FE. therefore inverſely AC, CB :: DF. FE.

3. *The Rectangle* ACB □ DFE. For ACq. ACB [b] :: AC. CB [c] :: DF. EF :: DFq. DFE. wherefore by inverſion ACq. DFq :: ACB. DFE. therefore being HCq □ DFq. [d] ſhall ACB be ⌭ DFE. [b] 1. 6. [c] *before.* [d] 10. 10.

4. ACq + CBq. □ DFq + FEq. For becauſe ACq. CBq [e] :: DFq. FEq. therefore by addition ACq + CBq. CBq :: DFq + FEq. FEq. therefore being CBq □ FEq, [f] ſhall alſo ACq + CBq. be □ DFq + FEq. [e] 22. 6. [f] 10. 10.

5. *Hence*, If AC ⌭ or ⌭ CB, [g] then likewiſe ſhall DE be ⌭ or ⌭ EF. [g] 10. 10.

PROP. LXVII.

A line DE, commenſurable in length to a binomial line (AC + CB) is it ſelf a binomial line, and of the ſame order. [a] *lem.* 66. 10.

Make AB. DE :: AC. DF. [b] then are AC, DF □. [c] and CB, FE □. whence being that AC and CB [d] are ρ ⌭, [e] thence DF, FE ρ ⌭. therefore DE is a binomial. But for that AC. CB [f] :: DF. FE. if AC □ or □ √ ACq—BCq, [g] then in like manner DF □ or □ √ DFq—FEq. alſo if AC □ or □ ρ propounded, [h] then ſhall DF [b] *hyp.* [c] *lem.* 66. 10. [d] *and ſch.* 12. 10. [e] 15. 10. [f] 12. 10. & 14. 10.

DF be □ or □ ρ propounded. But if CB □ or □ ρ. likewise FE □ or □ ρ. If both AC, CB, □ ρ. [f] then also both DF, FE, □ ρ. [g] That is, whatsoever binomial AB is, DE shall be of the same order. Which was to be demonstrated.

[f] *by def.* 48. 10. [g] 14. 10.

PROP. LXVIII.

A line DE commensurable in length to a bimedial line (AC + CB) is also a bimedial line, and of the same order.

Make AB. DE :: AC. DF, [b] therefore AC □ DF. and CB □ FE. therefore seeing AC and CB [c] are μ, [d] also DF and FE shall be μ, and for that AC [e] ⊟ CB, [f] therefore FD ⊟ FE. [g] therefore DE is 2 μ. Wherefore if the rectangle ACB be ρν. because DFE [h] □ ACB, [i] likewise DFE is ρν. and if that be μν, [k] this shall be μν too. [k] That is, whether AB be 1 bimed. or 2 bimed. DF shall be of the same order. Which was to be demonstrated.

[a] 12. 6. [b] *lem.* 66. 10. [c] *hyp.* [d] 24. 10. [e] 10. 10. [f] 38. 10. [g] *sch.* 12. 10 [h] 24. 10. [i] 38. *or* 39. 10.

PROP. LXIX.

A line DE commensurable to a major line (C + CB) is itself a major line.

Make AB. DE :: AC. DF. Because AC [a] ⊟ CB, [b] thence DF ⊟ FE. also ACq + CEq [a] is ρν. and so being DFq + FEq [b] □ ACq + CEq, [c] also DFq + FEq is ρν. lastly, the rectangle ACB [a] is μν. [d] therefore the rectangle DFE is μν. (because DFE is [b] □ ACB.) [e] wherefore DE is a *Major* line. Which was to be demonstrated.

[a] *hyp.* [b] *lem.* 66. 10. [c] *sch.* 12. 10. [d] 24. 10. [e] 40. 10.

PROP.

PROP. LXX.

A line DE commensurable to a line containing in power a rational and a medial rectangle (AC + CB) is a line containing in power a rational and a medial rectangle.

Again make AB . DE :: AC . DF. Because AC [a] ⊟ CB, [b] also DF ⊟ FE. likewise because ACq + CBq [a] is $\mu\nu$, [c] therefore DFq + FEq shall be $\mu\nu$. lastly, because the rectangle ACB [a] is $\text{'}\rho\nu$, [d] also DFE is $\text{'}\rho\nu$. Therefore DE contains in power $\rho\nu$ and $\mu\nu$. Which was to be demonstrated.

a *hyp.*
b *lem.* 66. 10.
c 24. 10.
d *sch.* 12. 10.
e 41. 10.

PROP. LXXI.

A line DE commensurable to a line containing two medial Rectangles in power (AC + CB) is also a line containing in power two medial Rectangles.

Divide DE, as in the *prec.* Because ACq [a] ⊏ CBq, [b] thence shall DFq be ⊏ FEq. also for that ACq + CBq [a] is $\mu\nu$, [c] shall DFq + FEq be also $\mu\nu$. And in like manner because ACB [a] is $\mu\nu$, [d] also DFE is $\mu\nu$. lastly, because ACq + CBq ⊏ ACB, [e] shall DFq + FEq be ⊏ DFE. [f] From whence it follows, that DE contains in power $\Omega\mu\alpha$. Which was to be demonstrated.

a *hyp.*
b *lem.* 16. 10.
c 24. 10.
d 24. 10.
e 14. 10.
f 42. 10.

PROP. LXXII.

If a rational rectangle A and a medial B, be composed together, these four irrational Lines will be made, either a binomial, or a first bimedial, or a major, or a line containing in power a rational and a medial rectangle.

Namely, if Hq = A + B, then H ſhall be one of the four lines which the Theorem mentions. For upon CD the propounded ῥ, [a] make the rectangle CE = A, and FI = B. [b] and ſo CI = Hq. Whereas then is A μν, likewiſe CE is μν. [c] therefore the latitude CF is ῥ ▭ CD. and becauſe B is μν, alſo FI ſhall be μν. [d] therefore FK is ῥ ▭ C D. [e] therefore CF, FK are ῥ ▭, and ſo the whole CK [f] is binom. wherefore if A ▭ B *i. e.* CE ▭ FI, [g] then CF ▭ FK. therefore if CF ▭ √CFq—FKq, [h] likewiſe CK ſhall be a 1. bin. and conſequently H = √CI [k] is a bin. If CF be ſuppoſed ▭ √CFq—FKq, [l] then ſhall CK be a 4. bin. wherefore H (√CI) [m] is a major line. But if A ▱ B, [n] then ſhall CF be ▱ FK, conſequently if FK ▭ √FKq—CFq, [n] then ſhall CK be a 2 bin. [o] wherefore H is a firſt 2 μ. laſtly, if FK ▭ √FKq—CFq. [p] then CK ſhall be a fifth binom. [q] whence H ſhall contain in power ῥν and μν. Which was to be demonſtrated.

[a] *cor.* 16. 6.
[b] 2 *ax.* 1.
[c] 21. 10.
[d] 23. 10.
[e] 13. 10.
[f] 37. 10.
[g] 1. 6.
[h] 1 *def.* 48. 10.
[k] 55. 10.
[l] 4 *def.* 48. 10.
[m] 58. 10.
[n] 2 *def.* 48. 10.
[o] 56. 10.
[p] 5. *def.* 48. 10.
[q] 59. 10.

PROP. LXXIII.

Fig. pr. 72. *If two medial rectangles A, B, incommenſurable to one another, be compoſed together, the two remaining irrational lines are made, either a ſecond bimedial, or a line containing in power two medial rectangles.*

As H containing in power A + B is one of the ſaid irrational lines. For upon CD propounded ῥ draw the rectangle CE = A, and FI = B. whence Hq = CI. Therefore becauſe CE and FI [a] are μα. [b] the latitudes CF, FK, ſhall be ῥ ▭ CD. alſo becauſe CE [a] ▭ FI, and CE. FI [c] :: CF. FK, [d] therefore CF ▭ FK. [e] there-

[a] *hyp.*
[b] 23. 10.
[c] 1. 6.
[d] 10. 10.
[e] there-

[c] therefore CK is a 3. bin. namely, if CF □ √ CFq—FKq. whence H = √ CL. [f] ſhall be a ſecond 2 μ. But if CF □ √ CFq—FKq, [g] then CK ſhall be a 6 binom. [h] and conſequently H contains in power 2 μα. Which was to be demonſtrated.

[c] 3 *def.* 48. 10.
[f] 57. 10.
[g] 6 *def.* 48. 10.
[h] 60. 10.

Here begin the Scuaries of Lines irrational by Subtraction.

PROP. LXXIV.

If from a rational line DF a rational line DE, commenſurable in power only to the whole DF, be taken away, the reſidue EF is irrational, and is called an Apotome or reſidual line.

For EFq [a] □ DEq; [b] but DEq is 'ρν; [c] therefore EF is ρ̂. Which was to be demonſtrated.

In numbers; let there be DF, 2. DE, √ 3. then EF ſhall be 2—√ 3.

[a] *lem.* 26. 10.
[b] *hyp.*
[c] 10. & 11 *def.* 10.

PROP. LXXV.

If from a medial line DF, a medial line DE commenſurable only in power to the whole DF, and comprehending with the whole DF a rational rectangle, be taken away, the Remainder EF is irrational, and is called a firſt reſidual line of a medial.

Fig. pr. 74.

For EFq [a] □ to the rectangle FDE. therefore ſeeing FDE [b] is 'ρν, [c] EF ſhall be ρ̂, Which was to be demonſtrated.

In numbers, let DF be υ√ 54, and DE υ√ 24. therefore EF is υ√ 54—υ√ 24.

[a] *ſch* 26. 10.
[b] *hyp.*
[c] 20. & 11. *def.* 10.

PROP. LXXVI.

Fig. pr. 74. *If from a medial line DF, a medial line DE be taken away being commensurable only in power to the whole DF, and comprehending together with the whole line DF a medial rectangle, the remainder EF is irrational, and is called a second residual of a medial line.*

Because DFq and DEq are $\mu\alpha$ $\square$, [a] therefore shall DFq + DEq be $\square$ DEq. [c] wherefore DFq + DEq is $\mu\nu$. also the rectangle FDE, [e] and so 2 FDE, [a] is $\mu\nu$. therefore EFq ([d] DFq + DEq — 2 FDE) [e] is $\rho\nu$. wherefore EF is ρ'. Which was to be demonstrated.

[a] *hyp.* [b] 16. 10. [c] 24. 10. [d] *cor.* 7. 2. [e] 27. 10.

In numbers, let DF be $u\sqrt{18}$, and DE $u\sqrt{8}$. then EF $u\sqrt{18} - u\sqrt{8}$.

PROP. LXXVII.

If from a right line AC be taken away a right line AB being incommensurable in power to the whole BC, and making with the whole AC that which is composed of their squares rational, and the rectangle contained under them medial, the remainder BC is irrational, and is called a Minor line.

For ACq + AEq [a] is $\rho\alpha$, but the rectangle ACB [a] is $\mu\alpha$, [b] therefore 2 CAB $\square$ ACq + AEq (2 [c] CAB + BCq.) [d] therefore ACq + ABq $\square$ BCq. [e] therefore BC is ρ'. Which was to be demonstrated.

[a] *hyp.* [b] *sch.* 12. 10. [c] 7. 2. [d] 27. 10. [e] 11. *def.* 10.

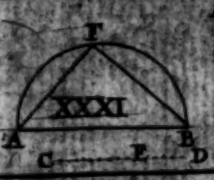
In numbers, let AC be $\sqrt{18} + \sqrt{108}$, AB $\sqrt{18} - \sqrt{108}$. then BC is $\sqrt{18} - \sqrt{108} - \sqrt{18} - \sqrt{108}$.

PROP.

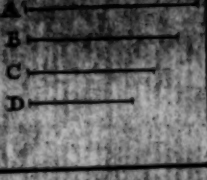
Euclid Lib. 10.

XXX

XXXI

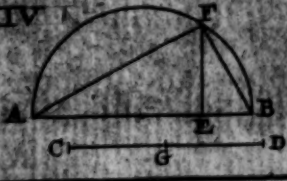


XXXII



XXXIII

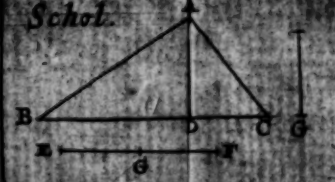
XXXIV



XXXV

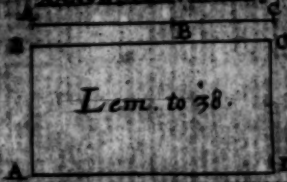
XXXVI

Schol.



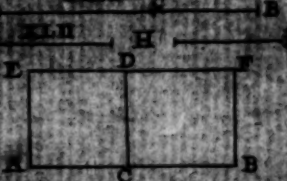
XXXVII. XXXVIII. XL.

Lem. to 38.



XXXIX

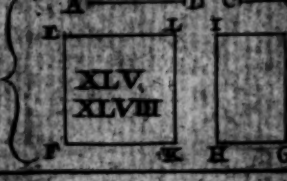
XLI

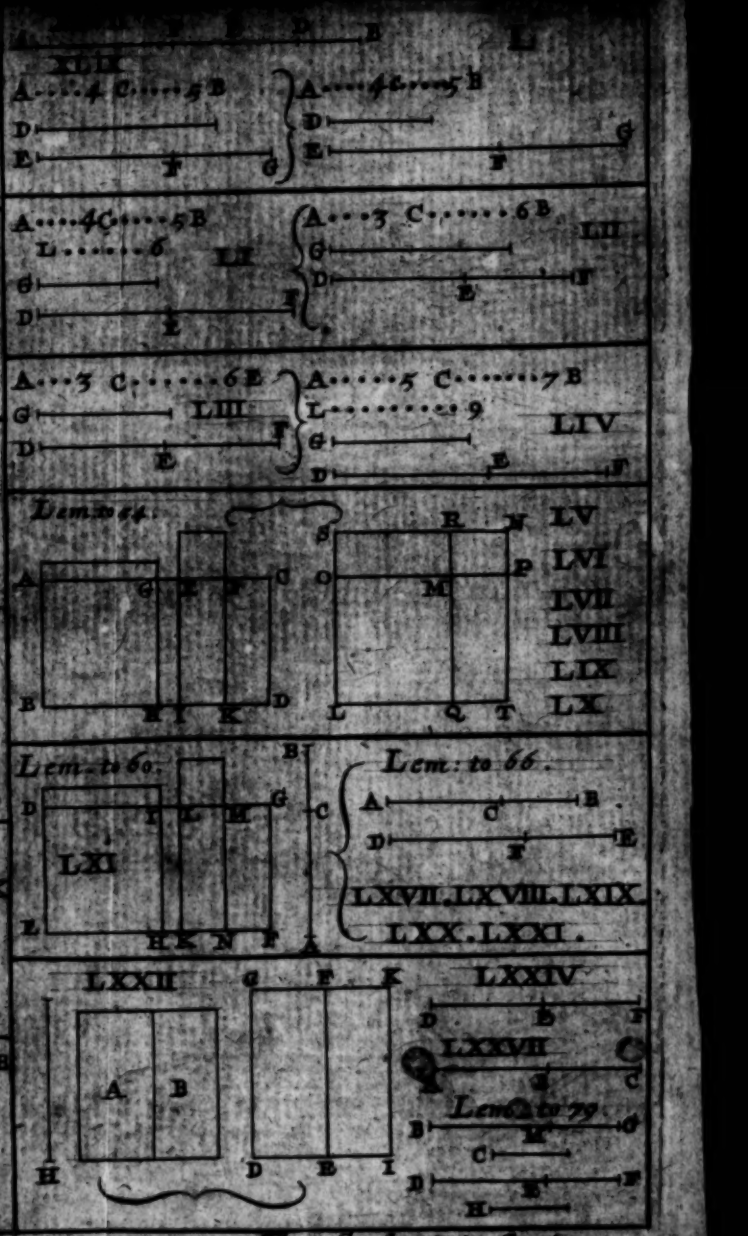


XLIII

XLIV

XLVI




XLII
LII
LIII
LIV
Lem. to 64.
LV
LVI
LVII
LVIII
LIX
LX
Lem. to 60.
LXI
Lem: to 66.
LXVII. LXVIII. LXIX.
LXX. LXXI.
LXXII
LXXIV
LXXVII
Lem. to 79.
Tho: Cross junior Sculp.

PROP. LXXVIII.

If from a right line DF be taken away a right line DE, being incommensurable in power to the whole line DF, and with the whole DF making that which is composed of their squares medial, and the rectangle contained under the same lines rational, the line remaining EF is irrational, and is called a line making a whole space medial with a rational space. Fig. pr. 7.

For 2 FDE [a] is ῥον. [b] and DFq + DEq is μεν. [c] therefore 2 FDE ⊓ DFq + DEq [d] (2 FDE + EFq) [e] therefore EF is ρ'. Which was to be demonstrated.

[a] hyp. & sch. 12.10.
[b] hyp.
[c] sch. 12. 10.
[d] 7. 2.
[e] sch. 12.10 and 11. def. 10.

In numbers, let DF be √: √216 + √72; DE √: √216 — √72. therefore EF is √: √216 + √72 — √: √216 — √72.

PROP. LXXIX.

If from a right line DF be taken away a right line DE, incommensurable in power to the whole DF, and which together with the whole makes that which is composed of their squares medial, and the Rectangle contained under them also medial and incommensurable to that which is composed of their squares, the remainder is irrational, and is called a line making a whole space medial with a medial space. Fig. pr. 74.

For 2 FDE, and FDq + DEq [a] are μα; [b] therefore EFq ([c] DFq + DEq — 2 FDE) is ρ'ν. [d] and so consequently EF is ρ'. Which was to be demonstrated.

[a] hyp. & 24. 10.
[b] 27. 10.
[c] cor. 7. 2.
[d] 11 def. 10

In numbers, let DF be √: √180 + √60. DE √: √180 — √60. then EF shall be √: √180 + √60 — √: √180 — √60.

Lemma.

Lemma.

If there be the same excess between the first magnitude BG and the second C (MG) as is between the third magnitude DF and the fourth H (EF;) then alternately, the same excess shall be between the first magnitude BG and the third DF, as is between the second C and the fourth H.

[a] *hyp.* For because that [a] to the equals BM, DE, are added the equals MG, EF, that is, C, H; the excess of the wholes BG, DF, [b] shall be equal to the excess of the parts added, C, H. Which was to be demonstrated.

[b] 15. ax. 1.

Corollarium.

Hence, four Magnitudes Arithmetically proportional, are alternately also Arithmetically proportional.

PROP. LXXX.

To an Apotome or residual line AB only one rational right line BC, being commensurable in power only to the whole AC, is congruent, or can be joyned.

If it be possible, let some other line BD be added to it; [a] then the rectangles ACB, ADB, [b] and so consequently the doubles of them are $\mu\alpha$. wherefore seeing ACq + BCq — 2 ACB [c] = ABq [c] = ADq + DBq — 2 ADB. therefore alternately ACq + BCq — : ADq + BDq [d] = 2 ACB — : 2 ADB. But ACq + BCq — : ADq + BDq [e] is $\rho\nu$. [f] therefore 2 ACB — : 2 ADB is $\rho\nu$. Which is Absurd.

[a] 22. 10.
[b] 22. 10.
[c] *cor.* 7. 2.
[d] *lem.* 79. 10
[e] *hyp. and* 27. 10.
[f] *sch.* 12. 10
[g] 27. 10.

PROP.

PROP. LXXXI.

To a first medial residual line AB only one medial right line BC, being commensurable only in power to the whole, and comprehending with the whole line a rational rectangle, can be joyned.

Conceive BD to be such a line as may be joyned to it; then because ACq and BCq, as well as ADq and BDq [a] are $\mu\alpha$ $\square$. [b] also ACq + BCq, & ADq + BDq shall be $\mu\alpha$. [c] but the rectangles ACB, ADB, [d] and so 2 ACB and 2 ADB are $\rho\alpha$. [e] therefore 2 ACB — : 2 ADB, [f] that is ACq + BCq — : ADq + BDq is $\rho\gamma$. [g] which is absurd.

[a] *hyp.*
[b] 10. *and* 24. 10.
[c] *hyp.*
[d] *scb.* 18. 10
[e] *scb.* 27. 10
[f] 7. 2. *and lem.* 79. 10.
[g] 27. 10.

PROP. LXXXII.

Unto a second medial residual line AB only one medial right line BC, commensurable only in power to the whole, and with it containing a medial rectangle, can be joyned.

If it be possible, let some other line BD be added to it; and upon EF ρ make the rectangle EG = ACq + BCq, as also the rectangle EL = ADq + BDq. likewise EI = ABq. Now 2 ACB + ABq = ACq + BCq = EG. therefore seeing EI = ABq, [a] also K G shall be = 2 ACB. moreover ACq and BCq [b] are $\mu\alpha$ $\square$. [c] therefore EG (ACq + BCq) is $\mu\nu$. [d] therefore the breadth EH is ρ $\square$ EF. [e] further, the rectangle ACB, [f] and so 2 ACB (KG) is $\mu\nu$. [g] therefore KH is also B, ρ $\square$ EF. lastly, because ACq + BCq (EG) [h] $\square$ 2 ACB (KG) and EG. KG [k] :: EH. KH, [l] therefore EH $\square$ HK. [l] therefore EK is a residual line, whereto K H is congruent. By the same reason also shall KM be congruent to the said EK.

[a] 4. 2. *and* 2. *ex.* 1.
[b] *hyp.*
[c] 24. 10.
[d] 23. 10.
[e] *hyp.*
[f] 24. 10.
[g] *lem.* 26. 10.
[h] 1. 6.
[k] 10. 10.
[l] 74. 10.

Which

Which is repugnant to the 80. Proposition of this Book.

PROP. LXXXIII.

To a Minor line AB only one right line BC can be joyned being incommensurable in power to the whole, and making together with the whole line that which is composed of their squares rational, and the rectangle which is contained under them medial.

Conceive any other BD to be congruent to it; Therefore whereas ACq + BCq and ADq + BDq [a] are ῥα. their excess (2 [b] ACB — : 2 ADB) [c] is ῥν. Which is absurd; because ACB and ADB, are μα by the Hyp.

a hyp.
b lem. 97. 10.
c sch. 27. 10
d 27. 10.

PROP. LXXXIV.

Unto a line (AB) making with a rational space a whole space medial only one right line BC can be joyned, being incommensurable in power to the whole, and making together with the whole that which is composed of their squares medial, and the Rectangle which is contained under them rational.

Suppose some other BD to be congruent also to it; [a] then the rectangles ACB, ADB, [b] and so 2 ACB and 2 ADB are ῥα. therefore 2 ACB — : 2 ADB, [c] that is, ACq + BCq — : ADq + BDq [d] is ῥν. Which is absurd, since ACq + BCq, and ADq + BDq are μα by the Hyp.

a hyp.
b sch. 12. 10
c lem. 79. 10.
d sch. 27. 10

PROP.

PROP. LXXIV.

To a line AB, which with a medial space makes a whole space medial, can be joyned only one right line BC, incommensurable in power to the whole, and making with the whole both that which is composed of their Squares medial, and the rectangle which is contained under them medial and incommensurable to that which is composed of their Squares.

Those things being supposed which are done and shewn in the 82 Prop. of this Book; it is clear that EH and KH are ⊐ EF. Besides, being that ACq + CBq, that is, the rectangle EG, [a] is ⊐ ACB, [b] and so EG ⊐ 2 ACB (KG;) and EG . KG = EH . KH; shall EH be ⊐ KH. therefore EK is a residual line, and the line congruent to it is KH. In like manner may KM be shewn to be congruent to the said residual EK, against the 80 Prop. of this Book.

[a] *hyp.*
[b] 14. 10.
[c] 1. 6.

Third Definitions.

A Rational line and a residual being propounded, if the whole be more in power than the line joyned to the residual, by the square of a right line commensurable unto it in length; then

I. If the whole be commensurable in length to the rational line propounded, it is called a first residual line.

II. But if the line adjoyned be commensurable in length to the rational line propounded, it is called a second residual line.

III. If

III. If neither the whole nor the line adjoyned be commenſurable in length to the rational line propounded, it is called a third reſidual line.

Moreover, If the whole be more in power than the line adjoyned by the ſquare of a right line incommenſurable to it in length, then

IV. If the whole be commenſurable in length to the rational line propounded, it is called a fourth reſidual line.

V. But if the line adjoyned be commenſurable in length to the rational line propounded, it is a fifth reſidual.

VI. If neither the whole nor the line adjoyned be commenſurable in length to the rational line propounded, it is termed a ſixth reſidual line.

PROP. LXXXVI. 87,88,89,90,91.

To find out a firſt, ſecond, third, fourth, fifth, and ſixth reſidual line.

Reſidual lines are found out by ſubducting the leſs names or parts of binomials from the greater. Ex. gr. Let $6 + \sqrt{20}$ be a firſt binom. then ſhall $6 - \sqrt{20}$ be a firſt reſidual. So that it is not neceſſary to repeat more concerning the finding of them out.

Lemma.

Let AC be a reſtangle contained under the right lines AB, AD. Let AD be drawn forth to E, and DE equally divided in F, and let the reſtangle AGE be = FEq. and the reſtangles AI, DK, FH, finiſhed. Then let the ſquare LM = AH be made,

made, and the ſquare NO = O1; and the lines NSR, OST, produced.

I ſay, 1. the rectangle AI = LM + NO = TOq + SOq. which appears by the conſtr.

2. *The rectangle* DK = LO. For becauſe the rectangle AGE [a] = FEq. [b] thence are AG, FE, GE, ∴ [c] and ſo AH, FI, GI ∴, [d] that is, LM, FI, NO, ∴; but LM, LO, NO, [d] are ∴; therefore FI = [e] LO [f] = DK = [g] NM.

3. *Hence*, AC = AI — DK — FI = LM + NO — LO — NM = TR.

4. [h] *It is manifeſt that* DF, FE, DE, *are* □.

5. *If* AE ⊐ DE, and AE □ √ AEq — DEq, [k] *then ſhall* AG, GE, AE, *be* □.

6. *Alſo, becauſe* AE [l] □ DE, [m] *thence ſhall* AE, FE, *be* □. [n] *and ſo* AI, FI, *that is*, LM + NO *and* LO *are* □.

7. *Becauſe* AG [*] □ GE, [o] *ſhall* AH, GI, *that is* LM, NO, *be* □.

8. *But becauſe* AE [l] □ DE, [p] *therefore ſhall* FE, GE, *be* □, [q] *and ſo the rectangle* FI □ GI, *that is*, LO □ NO. *wherefore ſeeing* LO. NO :: TS. SO. [q] *therefore ſhall* TS, SO, *be* □.

9. *If* AE *be put* □ √ AEq — DEq, [r] *then ſhall* AG, GE, AE, *be* □.

10. [s] *Wherefore the rectangles* AH, GI, *that is* TOq, SOq, *ſhall be* □.

a *conſtr.*
b 17. 6.
c 1. 6.
d *ſch.* 22. 6.
e 9. 5.
f 36. 1.
g 43. 1.
h 16. 10.
k 18. & 10 10.
l *hyp.*
m 13. 10.
n 1. 6. *and* 10. 10.
* *before.*
o 14. 10.
p 2. 6.
q 10. 10.
r 19. 10.
& 17. 10.
s 1. 6. *and* 10. 10.

PROP. XCII.

If a ſpace AC be contained under a rational line AB, and a firſt reſidual line AD (AE — DE) the right line TS, which containeth the ſpace AC in power, is a reſidual line.

Uſe the foregoing *Lemma* for a preparatory to the demonſtration of this Prop. Therefore TS = √ AC. Alſo AG, GE, AE, are □; there-

[a] *hyp.* [b] 12. 10. [c] 20. 10. [d] *lem.* 91. 10. [e] 74. 10.

therefore since AE □ [a] AB ῥ, [b] also AG and GE shall be □ AB, [c] therefore the rectangles AH and GI, that is, TOq and SOq are ῥα. [d] Likewise TO, SO, are ῥ □. [e] and consequently TS is a residual line. Which was to be demonstrated.

PROP. XCIII.

Fig. pr. 92.

If a space AC be contained under a rational line AB and a second residual AD (AE—DE) the right line TS, containing the space AC in power, is a first medial residual line.

Again, by the foregoing *Lemma*, AG, GE, AE are □. therefore [a] since AE is ῥ □ AB, [b] also AG, GE, shall be ῥ □ AB. [c] therefore the rectangles AH, GI, that is, TOq, SOq are μα, [d] likewise TO □ SO. Lastly, because DE [e] □ AB ῥ, [f] the right angle DI, and the half thereof DK or LO, that is TOS shall be μγ. [g] from whence it follows that TS (√ AC) is a first medial residual. Which was to be demonstrated.

[a] *hyp.* [b] 13. 10. [c] 22. 10. [d] *lem.* 74. 10. [e] *hyp.* [f] 20. 10. [g] 75. 10.

PROP. XCIV.

Fig. pr. 92.

If a space AC be contained under a rational line AB and a third residual AD (AE—DE) the right line TS containing in power the space AC is a second medial residual line.

As in the former, TO and SO are μ. Therefore because DE [a] is ῥ □ AB. [b] the rectangle DI, [c] and so DK, or TOS, shall be μγ. [d] therefore TS = √ AC is a second medial residual. Which was to be demonstrated.

[a] *hyp.* [b] 22. 10. [c] 24. 10. [d] 76. 10.

PROP.

PROP. XCV.

If a ſpace AC be contained under a rational line AB and a fourth reſidual AD (AE—DE) the right line TS containing the ſpace AC in power, is a Minor line. Fig. pr. 92.

As before, TO ▭ SO. Therefore becauſe AE[b] is ρ ▭ AB, [c] ſhall AI (TOq + SOq) be ρν, but, as before, the rectangle TOS is μν. [d] therefore TS = √AC is a *Minor* line. Which was to be demonſtrated.

[a] *lem.* 91. 10
[b] *hyp.*
[c] 20. 10.
[d] 77. 10.

PROP. XCVI.

If a ſpace AC be contained under a rational line AB and a fifth reſidual AD (AE—DE) the right line TS containing in power the ſpace AC, is a line which maketh with a rational ſpace the whole ſpace medial. Fig. pr. 92.

For again TO ▭ SO. Therefore ſince AE [a] is ρ ▭ AB, [b] alſo AI, that is TOq + SOq ſhall be μν. But, as in the 95. the rectangle TOS is ρν. [c] whence TS = √AC is a line which with ρν makes a whole μν. Which was to be demonſtrated.

[a] *hyp.*
[b] 22. 10.
[c] 78. 10.

PROP. XCVII.

If a ſpace AC be contained under a rational line AB, and a ſixth reſidual AD (AE—DE) the right line TS containing in power the ſpace AC, is a line making with a medial rectangle, a whole ſpace medial. Fig. pr. 92.

As often above, TO ▭ SO. alſo, as in 96, TOq + SOq is μν, but the rectangle TOS is ρν, as in the 94. [a] Laſtly, TOq + SOq ▭ TOS. [a] *lem.* 91. 10.

[b] 79. 10. TOS. [b] therefore TS = √AC is a line which with μγ makes a whole μγ. Which was to be demonstrated.

Lemma.

* cor. 16. 6. *Upon a right line DE* apply the rectangles DF = ABq, and DH = ACq, and IK = BCq. and let GL be bisected in M, and the line MN drawn parallel to GF.*

Then 1. *The rectangle* DK *is* = ACq + BCq. *as the Construction manifests.*

[a] *constr.* 2. *The rectangle* ACB = GN or MK. For
[b] 7. 2. DK [a] = ACq + BCq [b] = 2 ACB + AEq. but
[c] 3 *ax.* 1. ABq [a] = DF. therefore GK [c] = 2 ACB. and
[d] 7 *ax.* 1. [d] consequently GN or MK = ACB.

[e] 1. 6. 3. *The rectangle* DIL = MLq. For because ACq. ACB [e] :: ACB. BCq, that is DH. MK ::
[f] 17. 6. MK. IK. [e] thence is DI. ML :: ML. IL. [f] therefore DIL = MLq.

4. *If* AC *be taken* ⊟ BC, *then* DK *shall be* ⊟
[g] 16. 10. ACq. For ACq + BCq (DK) [g] ⊏ ACq.

5. *Likewise* DL ⊏ √DLq—GLq. For be-
[h] 10. 10. cause DH (ACq) ⊏ IK (BCq) [h] thence shall
[i] 18. 10. DI be ⊟ IL. [i] therefore √DLq—GLq ⊟ DL.

[l] *lem.* 26. 10 6. *Also* DL ⊏ GL. For ACq + BCq ⊟ 2
[m] 10. 10. ACB. that is, DK ⊟ GK. [m] therefore DL ⊏ GL.

[n] 19. 10. 7. *But if* AC *be taken* ⊟ BC, [n] *then* DL *shall be* ⊏ √DLq—GLq.

PROP. XCVIII.

The square of a residual line AB (AC — BC) applied to a rational line DE, makes the breadth DG a first residual line.

De

Do as is enjoyned in the *Lemma* next preceding. Then because AC, BC, [a] are 'ρ ⊐. [b] also DK (ACq + BCq) shall be ⊐ ACq. [c] therefore DK is 'ρν. [d] wherefore DL is 'ρ ⊐ DE. [e] likewise the rectangle GK (2 ACB) is μν. [f] therefore GL is 'ρ ⊐ DE [g] and consequently DL ⊐ GL. [h] But DLq ⊐ GLq. [k] therefore DG is a residual, [l] and that of the first order (because [m] AC ⊐ BC, and therefore DL ⊐ √ DLq — GLq.) Which was to be demonstrated.

[a] *hyp.*
[b] *le.* 97. 10.
[c] *sch.* 12. 10.
[d] 21. 10.
[e] 22. *and* 24. 10.
[f] 23. 10.
[g] 3. 10.
[h] *sch.* 12. 10
[k] 74. 10.
[l] 1. *def.* 85. 10.
[m] *le.* 97. 10.

PROP. XCIX.

The square of a first medial residual line AB (AC—BC) applied to a rational line DE, makes the breadth DG a second residual line. *Fig. pr* 98.

Supposing the foregoing *Lemma*; because AC and BC [a] are μ ⊐, [b] thence shall DK (ACq + BCq) be ⊐ ACq. [c] wherefore DK is μν. [d] therefore DL is 'ρ ⊐ DE. [e] also GK (2 ACB) is 'ρν. [f] therefore GL is 'ρ ⊐ DE; [g] wherefore DL ⊐ GL. [h] But DLq ⊐ GLq. [k] therefore DG is a residual line: and because DL is ⊐ √ DLq — GLq, [m] therefore shall DG be a second residual. Which was to be demonstrated.

[a] *hyp.*
[b] *lem.* 97. 10
[c] 24. 10.
[d] 23. 10.
[e] *hyp.* & *sch.* 12. 10.
[f] 21. 19.
[g] 13. 10.
[h] *sch.* 12. 10
[k] 74. 10.
[l] *le.* 97. 10.
[m] 2 *def.* 85. 10.

PROP. C.

The square of a second medial residual line AB (AC—BC) applied to a rational line DE, makes the breadth DG a third residual line. *Fig. pr.* 98.

Again DK is μν. [a] wherefore DL is 'ρ ⊐ DE. also DGK is μν. [a] whence GL is 'ρ ⊐ DE. [b] likewise DK ⊐ GK. [c] wherefore DL ⊐ GL. [d] but DLq ⊐ GLq. [e] therefore DG is a resi-

[a] 23. 10.
[b] *le.* 26. 10.
[c] 1. 6. *and* 10. 10.
[d] *sch.* 12. 10
[e] 74. 10.

f 3. def. 85. 10. e lem. 97. 10. residual line, and that of f the third order. e because DL □ √ DLq — GLq. Which was to be demonstrated.

PROP. CII.

Fig. pr. 98. *The square of a Minor line AB (AC — BC) applied to a rational line DE, makes the breadth DG a fourth residual.*

a 21. 10. * 89 p. b 23. 10. c 13. 10. d scb. 12. 10 e lem. 97. 10. f 4 def. 85. 10. As before, ACq + BCq, that is DK, is ρ. a therefore DL is ρ □ DE. but the rectangle ACB, and so GK (2 ACB) * is μγ. b wherefore GL is ρ □ DE. c therefore DL □ GL. d but DLq □ GLq. and because * ACq □ BCq, e thence shall DL be □ √ DLq — GLq. f therefore DG has the Conditions required to a fourth residual. Which was to be demonstrated.

PROP. CIII.

Fig. pr. 98. *The square of a line AB (AC — BC) which makes with a rational space the whole space medial, applied to a rational line DE, makes the breadth DG a fifth residual line.*

a 13. 10. b 21. 10. c 13. 10. d scb. 12. 10 e lem. 97. 10. f 5 def. 85. 10. For, as above, DK is μγ. a wherefore DL is ρ □ DE. also GK is ργ. b whence GL is ρ □ DE. c therefore DL □ GL. d but DLq □ GLq. Moreover DL e □ √ DLq — GLq. wherefore DG f is a fifth residual. Which was to be demonstrated.

PROP. CIV.

Fig. pr. 98. *The square of a line AB (AC — BC) making with a medial space the whole space medial, applied to a rational line DE, makes the breadth DG a sixth residual line.*

As

As above DK and GK are $\mu\alpha$; [a] wherefore DL and GL are ρ ⊏ DE. also DK [b] ⊏ GK. [c] whence DL ⊏ GL. [d] therefore DG is a residual. [b] and whereas ACq ⊏ BCq. and so DL ⊏ $\sqrt{}$ DLq—GLq. [e] therefore DG shall be a sixth residual. Which was to be demonstrated.

[a] 23. 10.
[b] *hyp. and lem.* 97. 10.
[c] 10. 10.
[d] 74. 10.
[e] 6 *def.* 85. 10.

PROP. CIV.

A right line DE commensurable in length to a residual AB (AC—BC) is it self also a residual, and of the same order.

Lemma.

Let AB.DE :: AC.DF. *and* AB ⊏ DE.

I say AC + BC ⊏ DF + EF. For AC.BC [a] :: DF.EF. therefore by addition AC + BC. BC :: DF + EF.EF. therefore by inversion AC + BC. DF + EF :: C.EF. [a] but BC ⊏ EF. [b] therefore AC + BC ⊏ DF + EF. Which was to be demonstrated.

[a] *lem.* 66. 10.
[b] 10. 10.

[a] Make AB.DE :: AC.DF. [b] therefore AC + BC ⊏ DF + EF. therefore seeing AC + BC [c] is a binomial, [d] DF + EF shall be a binomial too, and of the same order, [e] wherefore DF—EF is a residual of the same order with AC—BC. Which was to be demonstrated.

[a] 12. 6.
[b] *lem.* 103. 10.
[c] *hyp.*
[d] 67. 10.
[e] *by def.* 85. 10.

PROP. CV.

A right line DE commensurable to a medial residual line AB (AC—BC) is it self a medial residual, and of the same order. Fig. pr. 104

Again

Again, [a] make AB. DE :: AC. DF. [b] whence AC + BC ▭ DF + EF. [c] therefore DF + EF is a bimedial of the same order with AC + BC, [d] and consequently DF — EF shall be a medial residual of the same order with AC — BC. Which was to be demonstrated.

[a] 12. 6.
[b] *lem.* 103. 10.
[c] 68. 10.
[d] 75, *and* 76. 10.

PROP. CVI.

A right line DE commensurable to a Minor line AB (AC — BC) is it self also a Minor line.

Make AB. DE :: AC. DF. [a] then is AC + BC ▭ DF + EF. but AC + BC [b] is a *Major* line; [c] therefore DF + EF is also a *Major* line; [d] and consequently DF — EF is a *Minor* line. Which was to be demonstrated.

[a] *lem.* 103. 10.
[b] *hyp.*
[c] 69. 10.
[d] 77. 10.

PROP. CVII.

A right line DE commensurable to a line AB (AC — BC) which makes with a rational space the whole space medial, is it self also a line making with a rational space the whole space medial.

Fig. pr. 104

For, accordingly as in the former, we may shew DF + EF to contain in power *pr* and *μr*. [a] whence DF — EF is a line making, &c.

[a] 78. 10.

PROP. CVIII.

A right line DE commensurable to a line AB (AC — BC) which with a medial space makes the whole space medial, is it self a line making with a medial space the whole space medial.

Fig. pr. 106

For according to the preced. DF + EF shall contain in power 2 *μa*. [a] therefore DE — EF shall be, as in the Prop.

[a] 79. 10.

PROP.

PROP. CIX.

A medial rectangle B being taken from a rational rectangle A + B, the right line H which containeth in power the space remaining A, is one of these two irrational lines, viz. either a residual line, or a Minor line.

Upon CD ρ make the rectangles CI = A + B, and FI = B, whence CE [a] = A = Hq, wherefore because GI [b] is ρν, [c] therefore CK is ρ ▭ CD. but being FI [b] is μν, [d] shall FK be ρ ▭ CD. [e] whence CK ▭ FK. [f] therefore CF is a residual line. Wherefore if CK be ▭ √CKq—FKq, [g] then CF shall be a first residual; [h] therefore √CE (H) is a residual line. But if CK ▭ √CKq—FKq, [k] then CF shall be a fifth residual; and consequently H (√CE) [l] shall be a *Minor* line. Which was to be demonstrated.

[a] 3. *ax.* 1.
[b] *hyp. and constr.*
[c] 21. 10.
[d] 23. 10.
[e] 13. 10.
[f] 74. 10.
[g] 1 *def.* 85. 10.
[h] 92. 10.
[k] 4. *def.* 85. 10.
[l] 95. 10.

PROP. CX.

A rational rectangle B being taken away from a medial rectangle A + B, other two irrational lines are made, namely either a first medial residual line, or a line making with a rational space the whole space medial.

Fig. 109

Upon CD the propounded ρ make the rectangles CI = A + B, and FI = B. [a] whence CE = A = Hq. Therefore because CI [b] is μν, [c] shall CK be ρ ▭ CD. but because FI [b] is ρν. [d] thence FK ρ ▭ CD. [e] whence CK ▭ FK. [f] therefore CF is a residual, [g] and that a second. If CK ▭ √CKq—FKq, [h] then H (√CE) is a first medial residual. But if CK ▭ + CKq—FKq, [k] then shall CF be a fift residual;

[a] 3 *ax.* 1.
[b] *hyp. and constr.*
[c] 23. 10.
[d] 21. 10.
[e] 13. 10.
[f] 74. 10.
[g] 2 *def.* 85. 10.
[h] 93. 10.
[k] 5 *def.* 85. 10.

[1] 96. 10. sidual; and [1] consequently H (√CE) shall be a line making μν with ρν. Which was to be demonstrated.

PROP. CXI.

A medial space B being taken away from a medial space A + B, which is incommensurable to the whole A + B, the other two irrational lines are made; viz. either a second medial residual line, or a line making with a medial space the whole space medial.

Fig. pr. 109.

Upon CD ρ make the rectangles CE = A + B, and FI = B. [a] wherefore GE = A = Hq. Because therefore CI is μν, [b] thence GK is ρ □ CD. and in like manner FK ρ □ CD. Likewise because CI [c] = FI, [d] therefore GK □ FK. [e] wherefore GF is a residual, [f] namely a third. If CK □ √CKq—FKq; [g] whence H (√CE) shall be a second medial residual. but if CK □ √CKq—FKq [h] then shall CF be a sixth residual. [k] wherefore A shall be a line making μν with μν. Which was to be demonstrated.

[a] 3 ex. 1.
[b] 23. 10.
[c] hyp.
[d] 10. 10.
[e] 74. 10.
[f] 3 def. 85. 10.
[g] 94. 10.
[h] 6 def. 85. 10.
[k] 97. 10.

PROP. CXII.

A residual line A is not the same with a binomial line.

Upon BC propounded ρ make the rectangle CD = Aq. Therefore seeing A is a residual, [a] BD shall be a first residual, to which let DE be the line congruent or that may be adjoyned. [b] wherefore BE, DE, are ρ □. [c] and BE □ BC. If you conceive A to be a binomial, then BD is a first bin. whose names let be BF, FD, and let BF be □ FD. [d] therefore BF, FD,

[a] 98. 10.
[b] 74. 10.
[c] 1 def. 85. 10.
[d] 37. 10.

FD are ρ̇ ⊐; and BF [e] ⊏ BC. therefore since BC ⊏ BE, [f] shall BE be ⊏ BF. [g] and thence BE ⊏ FE. [h] therefore FE is ρ̇. Likewise because BE ⊏ DE, [k] shall FE be ⊏ DE. [l] wherefore FD is a residual, and so FD is ρ̇. but it was shewn ρ̇. which are repugnant. Therefore A is falsly conceived to be a binomial. Which was to be demonstrated.

[e] 1 *def*. 48. 10.
[f] 12. 10.
[g] *cor*. 16. 10
[h] *sch*. 12. 10
[k] 14. 10.
[l] 74. 10.

The Names of the 13 irrational lines differing one from another.

1. A Medial line.
2. A binomial line; of which there are six species.
3. A first bimedial line.
4. A second bimedial.
5. A Major line.
6. A line containing in power a rational superficies and a medial superficies.
7. A line containing in power two medial superficies.
8. A residual line: of which there are also six kinds.
9. A first medial residual line.
10. A second medial residual line.
11. A Minor line.
12. A line making with a rational superficies the whole superficies medial.
13. A line making with a medial superficies the whole superficies medial.

Being the differences of breadths do argue differences of right lines, whose squares are applied to some rational line, and it is demonstrated in the preced. Propositions that the breadths which arise from applying of the squares of these 13 lines do differ

differ one from another, it evidently follows that these 13 lines do also differ one from another.

PROP. CXIII.

The square of a rational line A applied to a binomial BC (BD + DC) makes the breadth EC a residual line, whose names EH, CH, are commensurable to the names BD, DC, of the binomial line, and in the same proportion (EH. BD :: CH. DC:) and moreover, the residual line EC which is made, is of the same order with BC the binomial.

Upon DC the less name [a] make the rectangle DF = Aq = BE. whence BC. CD [b] :: FC. CE. therefore by division, BD. DC :: FE. EC. And whereas BD [c] ⊏ DC, [d] thence FE shall be ⊏ EC. Take EG = EC, and make FG. GE :: EC. CH. Then EH and CH shall be the names of the residual EC, whereunto all is agreeable that is propounded in the Theorem. For being that by Addition FE. GE (EC) :: EH. CH. therefore FH. EH [e] :: EH. CH [f] :: FE. EC [f] :: BD. DC. wherefore since BD [g] ⊟ DC, [f] thence shall EH be ⊟ CH, [h] and FHq □ EHq. Therefore because FHq. EHq [k] :: FH. CH. [b] shall FH be □ CH, [l] and so FC □ CH. Moreover CD [g] is ῥ, and DF (Aq) [g] is ῥr. [m] therefore FC is ῥ, □ CD. whence also CH is ῥ □ CD. [n] therefore EH, CH are ῥ and ⊟, as before. [o] therefore EC is a residual line, to which CH may be joyned. Furthermore EH. CH [f] :: BD. DC. and so by Inversion EH. BD :: CH. DC. whence because CH [f] □ DC, [p] shall EH be □ BD. But suppose BD □ √ BDq — DCq. [q] then shall EH be □ √ EHq — CHq. Also if BD □ ῥ propounded, then shall EH be □ to the same ῥ.

[a] *cor.* 16. 6.
[b] 14. 6.
[c] *hyp.*
[d] 14. 5.
[e] 12. 5.
[f] *before.*
[g] *hyp.*
[h] 10. 10.
[k] *cor.* 20. 6
[l] 16. 10.
[m] 21. 10.
[n] *sch.* 12. 10
[o] 74. 10.
[p] 10. 10.
[q] 15. 10.

[f] that

[f] that is, if BC be a firſt binomial, [t] EC ſhall be a firſt reſidual. In like manner, if DC be to the □ propounded ῥ, [r] then is CH □ to the ſame ῥ. [u] that is, if BC be a ſecond binomial, [x] EC ſhall be a ſecond reſidual: and if this be a third binom. then that ſhall be a third reſidual, &c. But if BD be □ √ BDq — DCq, [y] then ſhall EH be □ √ EHq — CHq. therefore if BC be a 4, 5, or 6 binomial, EG ſhall be likewiſe a 4, 5, or 6 reſidual. Which was to be demonſtrated.

[r] 12. 10.
[f] 1 *def.* 48. 10.
[t] 1 *def.* 85. 10.
[u] 2 *def.* 48 10.
[x] 2 *def.* 85. 10.
[y] 15. 10.

PROP. CXIV.

The ſquare of a rational line A applied to a reſidual line BC (BD—CD) makes the breadth BE a binomial; whoſe names BE, GE, are commenſurable to the names BD, BC, of the reſidual line BC, and in the ſame proportion: and moreover, the binomial line which is made (BE) is of the ſame order with the reſidual line (BC.)

[a] Make the rectangle DF = Aq. and BF. FE [b] :: EG. GF. whence for that DF = Aq = CE, [c] therefore BD. BC :: BE. BF. therefore by converſion of proportion BD. CD :: BE. FE :: EG. GF :: [d] BG. EG. but BD [e] ⊟ CD. [f] therefore BG ⊟ GE. therefore becauſe BGq. GEq [g] :: BG. GF. [h] ſhall BG be □ GF. [k] and ſo BG □ BF. moreover, BD [e] is ῥ, and the rectangle DE (Aq) [e] is ῥν [l] therefore BF is ῥ □ BD. [m] therefore alſo BG is ῥ □ BD. [n] therefore BG, GE are ῥ ⊟. [o] wherefore BE is a binomial. Laſtly, becauſe BD. CD :: BG. GE. and inverſely BD. BG :: CD. GE. and ED □ BG. [p] thence ſhall CD be □ GE. therefore if CB be a firſt reſidual, BE ſhall be a firſt binomial, &c. as in the *prec.* therefore, &c.

[a] *cor.* 16. 6
[b] 12. 6.
[c] 14. 6.
[d] 19. 5.
[e] *hyp.*
[f] 10. 10.
[g] *cor.* 20. 6.
[h] 10. 10.
[k] *cor.* 16. 10.
[l] 21. 10.
[m] 12. 10.
[n] *ſcb.* 12 10
[o] 37. 10.
[p] 10. 10.

PROP. CXV.

If a ſpace AB be contained under a reſiduall line AC (CE—AE) and a binomial CB, whoſe names CD, DB are commenſurable to the names CE, AE, of the reſiduall line, and in the ſame proportion (CE. AE :: CD. DB) then the right line F which containeth in power that ſpace AB, is rational.

Let G be ρ. and make the rectangle CH = Gq; [a] then ſhall BH (HI—IB) be a reſiduall line, and HI [a] □ CD [b] □ CE. [a] and BI □ DB. [a] and HI. BI :: CD. DB [b] :: CE. EA. therefore by inverſion HI. CE :: BI. EA. [c] therefore BH. AC :: HI. CE :: BI. EA. wherefore ſince [d] HI □ CE. [e] thence BH □ AC. [f] therefore the rectangle HC □ BA. But HC (Gq) [b] is ρν. [g] therefore BA (Fq) is ρ: and conſequently F is ρ. Which was to be demonſtrated.

[a] 113. 10.
[b] *hyp.*
[c] 19. 5.
[d] 12. 10.
[e] 10. 10.
[f] 1. 6. *and* [g] *ſch.* 12. 10

Corollarium.

Hereby it appears that a rational ſuperficies may be contained under two irrational right lines.

PROP. CXVI.

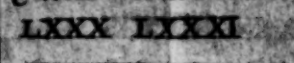
Of a medial line AB are produced infinite irrational lines BE, EF, &c. whereof none is of the ſame kind with any of the precedent.

Let AC be propounded ρ, and AD a rectangle contained under AC, AB. [a] therefore AD is ρν. Take BE = √AD. [b] then BE is ρ, and the ſame with none of the former. For no ſquare of any of the former being applied to ρ, makes the breadth medial. Let the rectangle

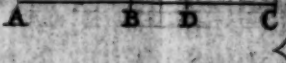
[a] *lem.* 38. 10.
[b] 11. 10.

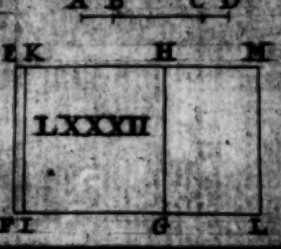
Euclid Lib. 10.

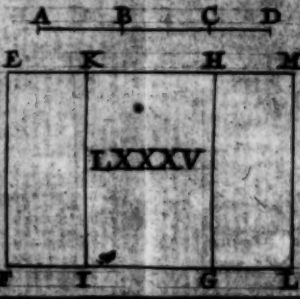
LXXX LXXXI

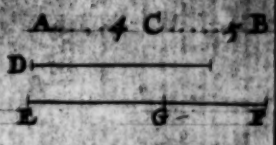


LXXXII LXXXIV



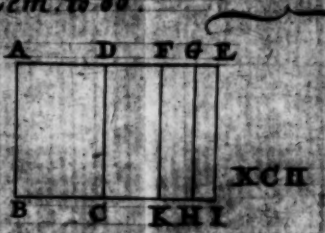


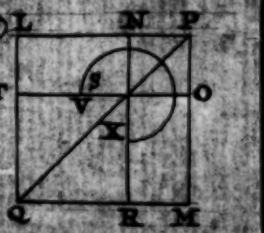




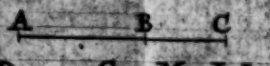
LXXXVI

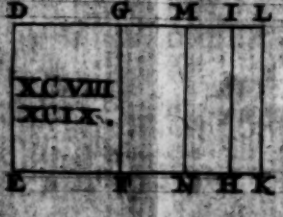
Lem. to 86.



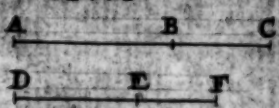


Lem. to 97.

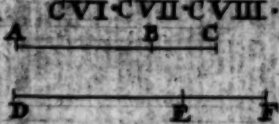


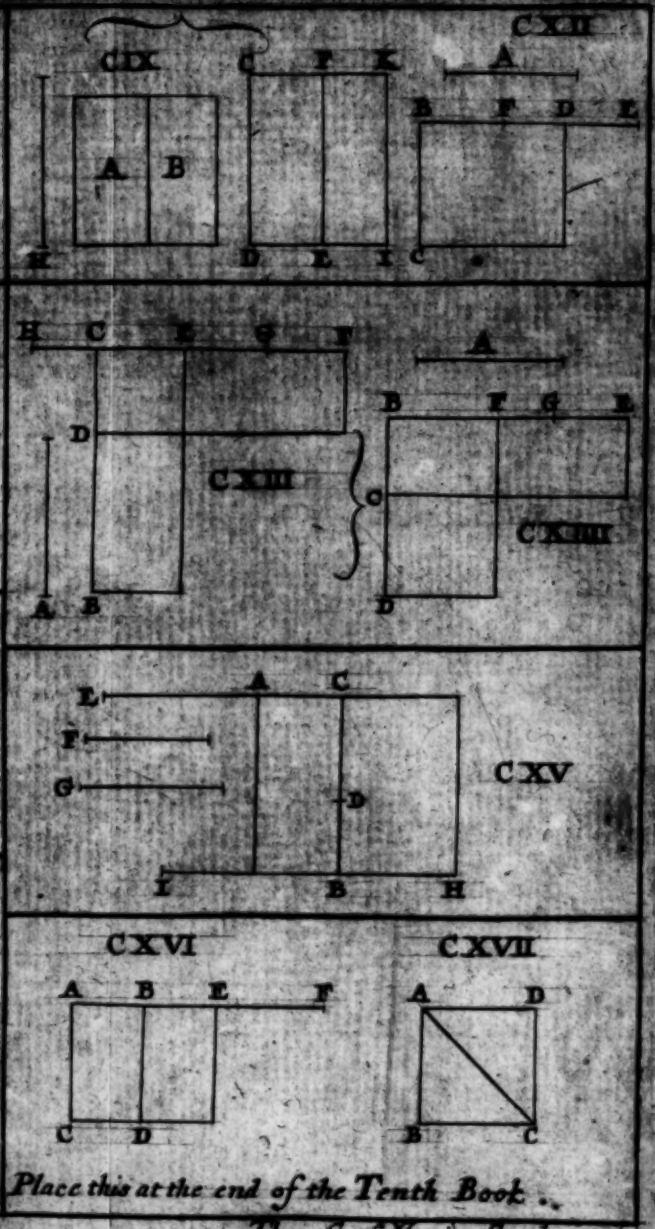


CIV. CV



CVI. CVII. CVIII.





Place this at the end of the Tenth Book.

The: Cross Junior Sculp.

angle DE be finiſhed, [a] then DE ſhall be ῥν. and [b] conſequently EF (√ DE) ſhall be ῥ, and not the ſame with any of the former: for no ſquare of the former being applied to ῥ, makes the latitude BE. therefore, &c.

PROP. CXVII.

Let it be required to ſhew that in ſquare figures BD, the diameter AC is incommenſurable in length to the ſide AB.

For ACq. ABq [a] :: 2.1 [b] :: not Q. Q. [c] therefore AC ⊓ AB. Which was to be demonſtrated. This Theorem was of great note with the ancient Philoſophers; ſo that he that underſtood it not was eſteemed by *Plato* undeſerving the name of a Man, but rather to be reckoned among Brutes.

[a] 47. 1.
[b] *cor.* 24. 8.
[c] 9. 10.

The end of the Tenth Book.

THE ELEVENTH BOOK OF Euclid's *Elements*.

Definitions.

I. A Solid is that which hath length, breadth, and thickneſs.

II. The term or extream of a ſolid is a ſuperficies.

III. A right line AB is perpendicular to a Plane CD, when it makes right Angles ABD, ABE, ABF, with all the right lines BD, BE, BF, that touch it, and are drawn in the ſaid Plane.

IV. A Plane AB, is perpendicular to a Plane CD, when the right lines FG, HK, drawn in one Plane AB to the line of common ſection of the two Planes EB, and making right Angles therewith, do alſo make right Angles with the other Plane CD.

V. The inclination of a right line AB to a Plane CD, is, when a perpendicular AE is drawn from A the higheſt point of that line AB to the plane CD, and another line EB drawn

drawn from the point E, which the perpendicular AE makes in the Plane CD, to the end B of the ſaid line AB which is in the ſame Plane, whereby the Angle is acute ABE which is contained under the inſiſting line AB, and the line drawn in the plane EB.

VI. The inclination of a Plane AB to a Plane CD, is an acute angle FHG contained under the right lines FH, GH, which being drawn in either of the Planes AB, CD to the ſame point H of the common ſection BE, make right Angles FHB, GHB, with the common ſection BE.

VII. Planes are ſaid to be inclined to other Planes in the ſame manner when the ſaid angles of Inclination are equal one to another.

VIII. Parallel Planes are thoſe which being prolonged never meet.

IX. Like ſolid Figures are ſuch as are contained under like Planes equal in number.

X. Equal and like ſolid Figures are ſuch as are contained under like Planes both in multitude and magnitude.

XI. A ſolid Angle is the inclination of more than two right lines which touch one another, and are not in the ſame Superficies.

Or thus:

A ſolid Angle is that which is contained under more than two plane Angles not being in the ſame Superficies, but conſiſting all at one point.

XII. A Pyramid is a ſolid Figure comprehended under divers Planes ſet upon one Plane, (which is the baſe of the Pyramid,) and gathered together to one point.

XIII. A Prism is a ſolid figure contained under planes, whereof the two oppoſite are equal, like, and parallel; but the others are parallelograms.

XIV. A Sphere is a ſolid figure made when the diameter of a ſemicircle abiding unmoved, the ſemicircle is turned round about, till it return to the ſame place from whence it began to be moved.

Corollarium.

Hence, all the rays drawn from the center to the ſuperficies of a ſphere, are equal amongſt themſelves.

XV. The Axis of a ſphere, is that fixed right line about which the ſemicircle is moved.

XVI. The Centre of a ſphere, is the ſame point with that of the ſemicircle.

XVII. The diameter of a ſphere, is a right line drawn through the center, and terminated on either ſide in the ſuperficies of the ſphere.

XVIII. A Cone is a figure made, when one ſide of a rectangled Triangle (*viz.* one of thoſe that contain the right angle) remaining fixed, the Triangle is turned round about till it return to the place from whence it firſt moved. And if the fixed right line be equal to the other which containeth the right Angle, then the Cone is a rectangled Cone: but if it be leſs, it is an obtuſe-angled Cone; if greater, an acute-angled Cone.

XIX. The Axis of a Cone is that fix'd line about which the Triangle is moved.

XX. The

XX. The Base of a Cone is the circle, which is described by the right line moved about.

XXI. A Cylinder is a figure made by the moving round of a right-angled Parallelogram, one of the sides thereof, (namely which contain the right angle) abiding fix'd, till the Parallelogram be turned about to the same place where it began to move.

XXII. The Axis of a Cylinder is that quiescent right line about which the Parallelogram is turned.

XXIII. And the Bases of a Cylinder are the circles which are described by the two opposite sides in their motion.

XXIV. Like Cones and Cylinders, are they both whose Axes and Diameters of their Bases are proportional.

XXV. A Cube is a solid figure contained under six equal squares.

XXVI. A Tetraedron is a solid figure contained under four equal and equilateral Triangles.

XXVII. An Octaedron is a solid figure contained under eight equal and equilateral Triangles.

XXVIII. A Dodecaedron is a solid figure contained under twelve equal, equilateral, and equiangular Pentagones.

XXIX. An Icosaedron is a solid figure contained under twenty equal and equilateral Triangles.

XXX. A Parallelepipedon is a solid figure contained under six quadrilateral figures, whereof those which are opposite are parallel.

XXXI. A ſolid figure is ſaid to be inſcribed in a ſolid figure, when all the Angles of the figure inſcribed are comprehended either within the Angles, or in the ſides, or in the planes of the Figure wherein it is inſcribed.

XXXII. Likewiſe a ſolid figure is then ſaid to be circumſcribed about a ſolid figure, when either the angles, or ſides, or planes of the circumſcribed figure touch all the angles of the figure which it contains.

PROP. I.

One part AC of a right line cannot be in a plane ſuperficies, and another part CB elevated upward.

Produce AC in the plane directly to F. If you conceive CB to be drawn ſtraight from AC, then two right lines AB, AF, have one common ſegment AC, [a] *which is impoſſible.*

[a] 10 ax. 1.

PROP. II.

If two right lines AB, CD, cut one another, they are in the ſame plane; And every Triangle DEB is in one and the ſame plane.

For imagine EFG, part of the Triangle DEB, to be in one plane, and the part FDGB to be in another, then EF part of the right line ED is in a plane, and the other part elevated upwards, [a] *which is abſurd.* Therefore the Triangle EDB is in one and the ſame plane; and ſo alſo are the right lines ED, EB; [a] wherefore the whole lines AB, DC, are in one plane. Which was to be demonſtrated.

[a] 1. 11.

PROP.

PROP. III.

If two planes AB, CD, cut one the other, their common ſection EF is a right line.

If EF the common ſection be not a right line, [a] then in the plane AB draw the right line EGF, [a] and in the plane CD the right line EHF. therefore two right lines EGF, EHF include a ſuperficies. [b] Which is abſurd.

[a] 1 *poſt* 1.
[b] 14 *ax.* 1.

PROP. IV.

If a right line EF be at right Angles erected upon two lines AB, CD, cutting one the other, at the common ſection E; it ſhall alſo be at right angles to the plane ACBD drawn by the ſaid lines.

Take EA, EC, EB, ED, equal one to the other, and joyn the right lines AC, CB, BD, AD. draw any right line GH through E, and joyn FA, FC, FD, FB, FG, FH. Becauſe AE is [a] = EB, and DE [a] = EC, and the Angle AED [b] = CEB, [c] therefore AD is = CB, [c] and likewiſe AC = DB. [d] therefore AD is parallel to CB, [d] and AC to DB. [e] wherefore the Angle GAE = EBH, and the angle AGE = EHB. But alſo AE [f] = EB. [g] therefore GE = EH, [g] and AG = BH. whence by reaſon of the right Angles, by the hyp. and ſo equall, at E, [h] the baſes FA, FC, FB, FD, are equal. Therefore the triangles ADF, FBC, are equilateral one to another. [k] and thence the angle DAB = ECF. Therefore in the triangles AGF, FBH, the ſides FG, FH, [l] are equal; and ſo by conſequence the triangles FEG and FEH are mutually equilateral. [m] therefore the angles FEG, FEH, are equal, and [n] ſo right an-

[a] *conſtr.*
[b] 15. 1.
[c] 4. 1.
[d] *ſch.* 34. 1.
[e] 29. 1.
[f] *conſtr.*
[g] 26. 1.
[h] 4. 1.
[k] 8. 1.
[l] 4. 1.
[m] 8. 1.
[n] 10 *def.* 1.

angles. In like manner, FE makes right angles with all the lines drawn through E in the plane ADBC, [b] and is therefore perpendicular to the said plane.

[b] 2 def. 11.

PROP. V.

If a right line AB be erected perpendicular to three right lines AC, AD, AE, touching one the other at the common section, those three lines are in the same plane.

For AC, AD, [a] are in one plane FC; [a] and AD, AE, are in one plane BE: which if you conceive to be several planes, then let their intersection [b] be the right line AG; therefore because BA by the Hypoth. is perpendicular to the right lines AC, AD, [c] and so to the plane FC, [d] it is also perpendicular to the right line AG. therefore (since [a] that AB is in the same plane with AC, AE) the angles BAG, BAE, are right angles, and consequently equal, the part and the whole. Which is absurd.

[a] 2. 11.
[b] 3. 11.
[c] 4. 11.
[d] 3 def. 11.

PROP. VI.

If two right lines AB, DC, be erected perpendicular to one and the same plane EF, those right lines AB, DC, are parallel one to the other.

Draw AD, whereunto let DG = AB be perpendicular in the plane EF, and joyn BD, BG, AG. Being in the triangles BAD, ADG, the angles DAB, ADG [a] are right angles, and AB [b] = DG, and AD is common, [c] therefore BD is = AG. whence in the triangles AGB, BGD, equilateral one to the other, the angle BAG is [d] = BDG; of which being BAG is a right

[a] hyp.
[b] constr.
[c] 4. 1.
[d] 8. 1.

right angle, BDG must be so also. but the angle GDC is supposed right, therefore the right line GD is perpendicular to the three lines DA, DB, CD. [e] which are therefore in the same plane [f] wherein AB is. Wherefore since AB and CD are in the same plane, and the internal angles BAD, CDA, are right angles, [g] AB and CD shall be Parallels. Which was to be demonstrated.

[e] 5. 11.
[f] 2. 11.
[g] 28. 1.

PROP. VII.

If there be two parallel right lines AB, CD, and any points E, F, be taken in both of them, the line EF which is joyned at those points, is in the same plane with the parallels ABCD.

Let the plane, in which AB, CD, are, be cut by another plane at the points E, F. then if EF is not in the plane ABCD, it shall not be the common section. Therefore let EGF be the common section; which [a] then is a right line. therefore two right lines EF, EGF, include a superficies. [b] which is absurd.

[a] 3. 11.
[b] 14. *ax.* 1.

PROP. VIII.

If there be two parallel right lines AB, CD, whereof one AB is perpendicular to a plane EF. then the other CD shall be perpendicular to the same plane EF.

The preparation and demonstration of the sixth of this Book being transferr'd hither; the angles GDA, and GDB are right angles. [a] therefore GD is perpendicular to the plane, wherein are AD, DB ([b] in which also AB, CD, are.) [c] therefore GD is perpendicular to CD. but the angle CDA is also [d] a right angle.

[a] 4. 11.
[b] 7. 11.
[c] 3 *def.* 11.
[d] 29. 1.

[e] there-

[c] 4. 11. [c] therefore CD is perpendicular to the plane EF. Which was to be demonstrated.

PROP. IX.

Right lines (AB, CD) which are parallel to the same right line EF, but not in the same plane with it, are also parallel one to the other.

In the plane of the parallels AB, EF, draw HG perpendicular to EF; also in the plane of the parallels EF, CD, draw IG perpendicular to EF. [a] therefore EG is perpendicular to the plane wherein HG, GI are; and AH, GI, are perpendicular to the same plane. [c] therefore AH and CI are parallels. Which was to be demonstrated.

[a] 4. 11.
[b] 8. 11.
[c] 6. 11.

PROP. X.

If two right lines AB, AC, touching one another, be parallel to two other right lines ED, DF, touching one another, and not being in the same plane, those right lines contain equal Angles, BAC, EDF.

Let AB, AC, DE, DF, be equal one to the other, and draw AD, EC, EF, BE, CF. Being AB, DE, [a] are parallels and equal, [b] also BE, AD, are parallels and equal. In like manner, CF, AD, are parallels and equal. [c] Therefore also BE, FC are parallels and equal. [d] Therefore BC, EF, are equal. Wherefore since the Triangles BAC, EDF, are of equal sides one to the other, the angles BAC, EDF [e] shall be equal. Which was to be demonstrated.

[a] *hyp. and constr.*
[b] 33. 1.
[c] 2 *ax.* 1. & 30. 1.
[d] 33. 1.
[e] 8. 1.

PROP.

PROP. XI.

From a point given on high A to draw a right line AI perpendicular to a plane below BC.

In the plane BC draw any line DE, to which from the point A [a] draw the perpendicular AF, and [b] likewise FH in the plane BC cutting the said line DE at F; [a] then let fall AI perpendicular to FH. Which AI shall be perpendicular to the plane BC. [a] 12. I. [b] 11. I.

For through I [c] let KIL be drawn parallel to DE. Because DE [d] is perpendicular to AF, and FH, [e] therefore DE shall be perpendicular to the plane IFA. and so also KL [f] is perpendicular to the same plane. [g] therefore the angle KIA is a right angle, but the angle AIF is also [h] a right angle, [i] therefore AI is perpendicular to the plane BC. Which was to be done. [c] 31. I. [d] *constr.* [e] 4. 11. [f] 8. 11. [g] 3 *def.* 11. [h] *constr.* [i] 4. 11.

PROP. XII.

In a plane given BC, at a point given therein A, to erect a perpendicular line AF.

From some point without the plane D, [a] draw DE perpendicular to the said plane BC. and joyning the Points A, E, by a line AE, [b] draw AF parallel to DE. [c] it is apparent that AF is perpendicular to the plane BC. Which was to be done. [a] 11. 11. [b] 31. I. [c] 8. 11.

This and the preceding Problem are practically performed by applying two Squares to the Point given, as appears by 4. 11.

PROP.

PROP. XIII.

At a point given C in a plane given AB, two right lines CD, CE, cannot be erected perpendicular on the same side.

a 6. 11. For both CD, and CE, a should then be perpendicular to the plane AB, and consequently Parallels; which is repugnant to the definition of parallel lines.

PROP. XIV.

Planes CD, FE, to which the same right line AB is perpendicular, are parallel.

If you deny this; then let the planes CD, FE, meet, so that their common section be the right line GH, in which take any point I, draw to it the right lines IA, IB, in the said planes. whereby in the triangle IAB, two angles IAB, IBA, a are right angles. b Which is absurd.

a *hyp. and* 3 *def.* 11.
b 17. 1.

PROP. XV.

If two right lines AB, AC, touching one the other, be parallel to two other right lines DE, DF, touching one the other, and not being in the same plane with them, the plane BAC, EDF, drawn by those right lines, are parallel one to the other.

From A a draw AG perpendicular to the plane EF, b and let GH, GI, be parallel to DE, DF, c these also shall be parallel to AB, AC. Therefore since the angles IGA, HGA, d are right angles, also CAG, BAG, e shall be right angles. f therefore GA is perpendicular to

a 11. 11.
b 31. 1.
c 30. 1.
d 3 *def.* 11.
e 29. 1.
f 4. 11.

to the plane BC; [g] but the same is perpendicular to the plane EF. [h] therefore the planes BC, EF, are parallel. Which was to be demonstrated.

[g] constr.
[h] 14. 11.

PROP. XVI.

If two parallel planes AB, CD, be cut by some other plane HEIGF, their common sections EH, GF, are parallel one to the other.

For if they be conceived to be otherwise; being in the same plane that cuts them, they will meet somewhere, if produced: suppose in I; wherefore since the whole lines HEI, FGI, [a] are in the planes AD, CD, being produced, the planes also shall meet. Contrary to the Hyp.

[a] 1. 11.

PROP. XVII.

If two right lines ALB, CMD, be cut by parallel planes EF, GH, IK; they shall be cut proportionally. (AL. LB :: CM. MD.)

Let the right lines AC, BD, be drawn in the planes EF, IK; as also AD passing through the plane GH in the point N. and joyn NL. LM. the planes of the triangles ADC, ADB, make the sections BD, LN, and AC, NM, [a] parallels. Therefore AL . LB :: AN . ND [b] :: CM . MD. Which was to be demonstrated.

[a] 16. 11.
[b] 2. 6.

PROP. XVIII.

If a right line AB be perpendicular to some plane CD, all the planes extended by that right line AB (EF, &c.) shall be perpendicular to the same plane CD.

Let

Let there be ſome plane EF drawn by AB, making the ſection EG with the plane CD; from ſome point whereof H, [a] draw HI parallel to AB in the plane EF; [b] then ſhall HI be perpendicular to the plane CD, and ſo likewiſe any other lines that are perpendicular to EG. [c] therefore the plane EF is perpendicular to the plane CD; and by the ſame reaſon any other planes drawn by AB ſhall be perpendicular to EF. Which was to be demonſtrated.

[a] 31. I.
[b] 8. 11.
[c] 4 *def.* 11.

PROP. XIX.

If two planes AB, CD, cutting one the other, be perpendicular to ſome plane GH, their line of common ſection EF ſhall be perpendicular to the ſame plane (GH.)

Becauſe the planes AB, CD, are taken perpendicular to the plane GH, it appears by 4 def. 11. that out of the point F there may be drawn in both planes AB, CD, a perpendicular to the plane GH. which ſhall be [a] but one; and therefore the common ſection of the ſaid planes. Which was to be demonſtrated.

[a] 13. 11.

PROP. XX.

If a ſolid angle ABCD be contained under three plane angles, BAD, DAC, BAC, any two of them, howſoever taken, are greater than the third.

If the three angles are equal, the aſſertion is evident; if unequal, then let the greateſt be BAC; from whence [a] take away BAE = BAD, and make AD = AE; and alſo draw BEC, BD, DC.

[a] 23. I.

Be-

Because the side BA is common, and AD [b] = AE, and the angle BAE [b] = BAD, [c] thence is BE = BD. but BD + DC is [d] ⊏ BC, [e] therefore DC ⊏ EC. Wherefore since AD [b] = AE, and the side AC is common, and DC ⊏ EC, [f] the angle CAD shall be ⊏ EAC. [g] therefore the angle BAD + CAD ⊏ BAC. Which was to be demonstrated.

[b] *constr.* [c] 4. I. [d] 20. I. [e] 5. *ax.* I. [f] 25. I. [g] 4. *ax.* I.

PROP. XXI.

Every solid Angle A is contained under less angles than four plane right angles.

For let a plane any-wise cutting the sides of the solid angle A make a many-sided figure BCDE, and as many triangles ABC, ACD, ADE, AEB. I denote all the angles of the polygone by X, and I term the sum of the angles at the bases of the triangles Y. wherefore X + 4 Right ang. [a] = Y + A. but being that, (of the angles at B) [b] the angle ABE + ABC is ⊏ CBE, and the same is true also of the angles at C, at D, and at E, [c] it is manifest that Y is ⊏ X, and consequently A shall be ⊐ 4 right ang. Which was to be demonstrated.

[a] 32. I. *and* [b] 32. I. [c] 20. II. [d] 5. *ax.* I.

PROP. XXII.

If there be three plane angles A, B, HCI, whereof two howsoever taken are greater than the third, and the right lines which contain them be equal AD, AE, FB, &c. then of the right lines DE, FG, HI, coupling those equal right lines together, it is possible to make a triangle.

A triangle may be [a] made of them, if any two be greater than the third: but they are so. For [b] make the angle HCK = B, and CK = CH,

[a] 22. I. [b] 23. I.

= CH, and draw HK, IK. [e] thence KH = FG. and because the Angle KCI [d] ∟ A. [e] therefore KI ∟ DE, but KI [f] ∟ HI + KH (FG.) therefore DE ∟ HI + FG. By the like argument any two may be proved greater than the third; and consequently [a] it is possible to make a triangle of them. Which was to be demonstrated.

e 4. 1.
d hyp.
e 24. 1.
f 20. 1.

PROP. XXIII.

To make a solid angle MHIK of three plane angles A, B, C, whereof two howsoever taken are greater than the third. * *But it is necessary that those three Angles be less than four right Angles.*

* 21. 11.

Make AD, AE, BE, BF, CE, CG, equal one to the other; and of the subtended lines DE, EF, FG (that is, of the equal lines HI, IK, KH) [a] make the triangle HKI, about which [b] describe the circle LHKI. [c] But because AD is ∟ HL, [e] let ADq be = HLq + LMq. [d] and let LM be perpendicular to the plane of the circle HKI, and draw HM, KM, IM. wherefore since the angle HLM [e] is a right angle, [f] thence is MHq = HLq + LMq [g] = ADq. therefore MH = AD. By the same reason MK, MI, AD (that is AE, EB, &c.) are equal; therefore since HM = AD, and MI = AE, and DE [h] = HI, [k] the angle A shall be = HMI, [k] as likewise the angle IMK = B, [k] and the angle HMK = C, wherefore a solid angle is made at M of the three given plane Angles. Which was to be done. AD is assumed to be ∟ HL. But this is manifest. For if AD be = or ∟ HL, then is the angle A [l] = [m] or ∟ HLI. In like manner shall B be equal or ∟ HLK, and

a 22. 11. & 21. 1.
b 5. 4.
* See Clavius.
c sch. 47. 1.
d 12. 11.
e 3 def. 11.
f 47. 1.
g constr.
h constr.
k 8. 1.
l constr. and 8. 1.
m 21. 1.

C =

C = or ⌐ KLI. wherefore A + B + C shall either equal or exceed four right angles, Contrary to the Hypoth. therefore rather let AD be HL. Which was to be demonstrated. * 4 cor. 13. 1.

P. R. Q. P. XXIV.

If a solid AB be contained under six allel planes, the opposite planes thereof (AG, DB, &c.) are like and equal parallelograms.

The plane AC cutting the parallel planes AG, DB, [a] makes the sections AH, DC, parallels, and by the same reason AD, HC, are parallels. Therefore ADCH is a pgr. By the like argument the other planes of the parallelepipedon are [b] pgrs. wherefore being AF is parallel to HG, and AD to HC, [c] the angle FAD shall be = CGH. therefore because AE [d] = HG, and AD [d] = HC, and so AF. AD :: HG. HC, the triangles FAD, GHC, [e] are like and [f] equal; and consequently the pgrs. AE, HB, are like and [g] equal, and the same may be shewn of the rest opposite planes. therefore, &c. [a] 16. 11. [b] 35. *def.* 11. [c] 10. 11. [d] 34. 1. [e] 7. 5. [f] 6. 6. [g] 4. 1. [h] 6 *ax.* 1.

P. R. Q. P. XXV.

If a solid Parallelepipedon ABCD be cut by a plane EF parallel to the opposite planes AD, BC; then as the base AH is to the base BH, so shall solid AHD be to solid BHC.

Conceive the Parallelepipedon to be extended on either side, and take AI = AE, and BK = EB, and put the planes IQ, KP, [a] parallel to the planes AD, BC; then the pgrs. IM, AH, and [a] DL, DG, [b] and IQ, AD, EF, &c. are [a] like and equal. [c] wherefore the Parallelepipedon AQ is = AF; and by the same reason [a] 36. 1. *and* 1 *def.* 6. [b] 24. 11. [c] 10. *def.* 11

reason the Parallelepipedon BP = BF. therefore the ſolids IF, EP, are as multiplex of the ſolids AF, EC, as the baſes IH, KH, are of the baſes AH, BH. And if the baſis TH be
d 24. 11. & 9 def. 11. ▱, =, ▱ KH, d likewiſe ſhall the ſolid IE be ▱, =, ▱ EP. e conſequently AH. BH :: AF.
e 6 def. 5. EC. Which was to be demonſtrated.

The ſame may be accommodated to all ſorts of Priſmes, whence

Corollaries:

If any Priſm whatſoever be cut by a plane parallel to the oppoſite planes, the ſection ſhall be a figure equal and like to the oppoſite planes.

PROP. XXVI.

Upon a right line given AB, and at a point given in it A, to make a ſolid angle AHIL equal to a ſolid angle given CDEF.

a 11. 11. From ſome point F a draw FG perpendicular to the plane DCE, and draw the right lines DF, FE, EG, GD, CG. Make AH = CD, and the angle HAI = DCE, and AI = CE; and in the plane HAI make the angle HAK = DCG, and AK = CG. then erect KL perpendicular to the plane HAI, and let KL be = GF. and draw AL: then AHIL ſhall be a ſolid angle equal to that given CDEF. For the conſtruction of this does wholly reſemble the framing of that, as may eaſily appear to any that examine it.

PROP.

PROP. XXVII.

Upon a right line given AB to describe a parallelepipedon AK, like, and in like manner situate, with a solid parallelepipedon given CD.

Of the plane angles, BAH, HAI, BAI, which are equal to FCE, ECG, FCG, [a] make the solid angle A equal to the solid angle C. also [b] make FC. CE :: BA. AH. [b] and CE. CG :: AH. AI (c whence of equality FC. CG :: BA. AI) and finish the parallelepipedon AK, which shall be like to that which is given. [a] 26. 11. [b] 12. 6. [c] 22. 5.

For by the construction, the Pgr. [d] BH is like to FE, and [d] HI to EG, and [d] BI to FG, and [e] so the opposites of these to the opposites of them: therefore the six planes of the solid AK are like to the six planes of the solid CD, [f] and consequently AK, CD, are like solids. Which was to be done. [d] 1 *def.* 6. [e] 14. 11. [f] 9 *def.* 11.

PROP. XXVIII.

If a solid parallelepipedon AB be cut by a plane FGCD drawn by the diagonal lines DF, CG, of the opposite planes AE, HB, that solid AB shall be equally bisected by the plane FGCD.

For because DC, FG, are [a] equal and parallels, [b] the plane FGCD is a Pgr. and being [a] the Pgrs. AE, HB, are equal and like, [b] also the triangles AFD, HGC, CGB, DFE, are equal and like. But the Pgrs. AC, AG, are equal and like to FB and FD. therefore all the planes of the prism FGCDAH are equal and like to all the planes of the prism FGCDEB, and [c] consequently this prism is equal to that. Which was to be demonstrated. [a] 24. 11. [b] 34. 1. [c] 9 *def.* 11.

PROP. XXIX.

Solid Parallelepipedons AGHEFBCD, AGHEMLKI, being conſtituted upon the ſame baſe AGHE, and in the ſame height, whoſe inſiſting lines AF, AM, are placed in the ſame right lines AG, FL, are equal one to the other.

* *i. e. between the parallel planes AGHE, FLKD. and ſo underſtand it in the foll.*

For [a] if from the equal priſms AFMEDI, GBLHCK, the common priſm NBMPCI be taken away, and the ſolid AGNEHR be added, the Parallelepiped. AGHEFBCD ſhall be = AGHEMLKI. Which was to be demonſtrated.

[a] 10 *def.* 11 *and* 35. 1. [b] 3. *and* 2. *ax.* 1.

PROP. XXX.

Solid parallelepipedons ADBCHEFG, ADCBIMLK, being conſtituted upon the ſame baſe ADBC, and in the ſame height, whoſe inſiſting lines AH, AI, are not placed in the ſame right lines, are equal one to the other.

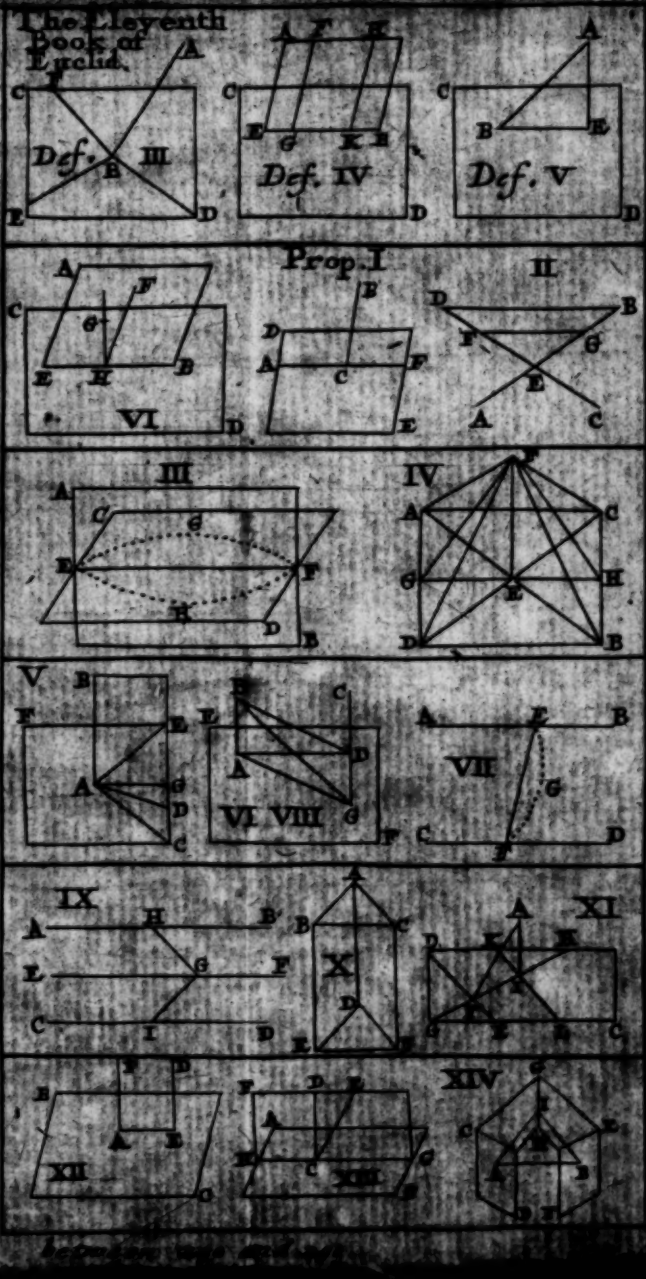
For produce the right lines HEO, GFN, and LMO, KIP; and draw AP, DO, BQ, CN. [a] then ſhall DC, AB, HG, EF, PQ, ON, be as well equal and parallel one to the other as AD, HE, GF, BC, KL, IM, QN, PO. [b] wherefore the parallelepipedon ADCBPONQ ſhall be equal to either parallelepipedon ADCBHEFG, ADCBIMLK; and [c] conſequently theſe two are equal one to the other. Which was to be demonſtrated.

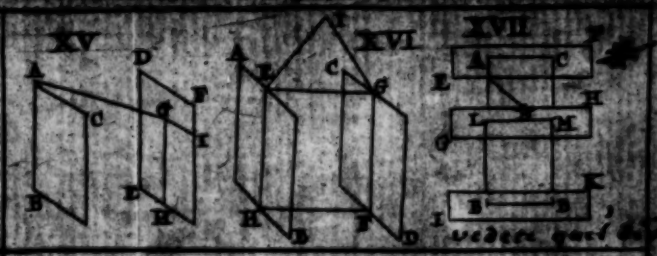
[a] 34. 1. [b] 29. 11. [c] 1 *ax.* 1.

PROP. XXXI.

Solid parallelepipedons ALEKGMBI, CPωOHQDN, being conſtituted upon equal baſes ALEK,

The Eleventh Book of Euclid.

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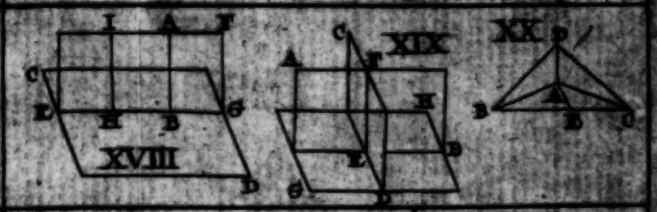
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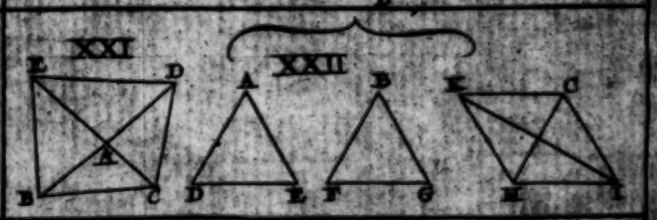
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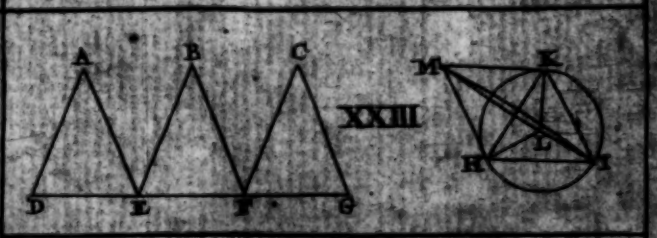
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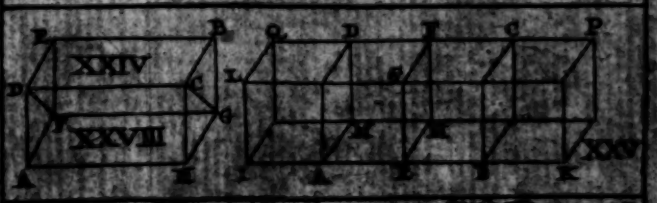
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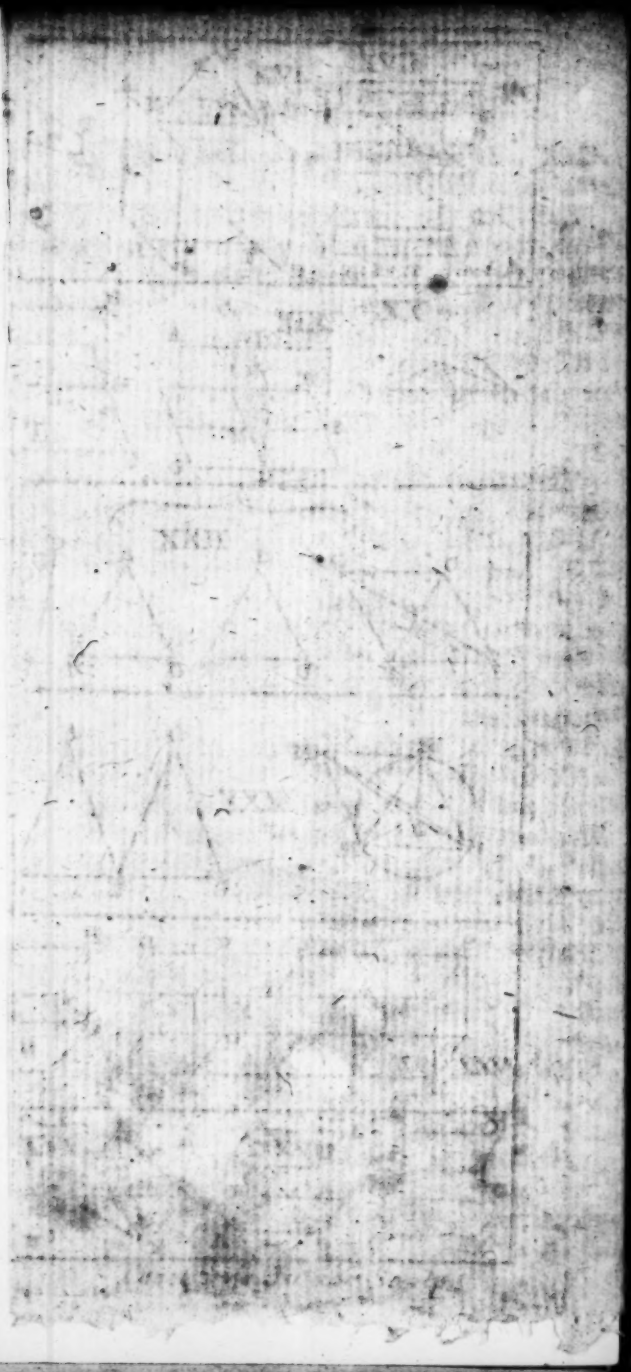

ALEK, CPωO, and * *in the ſame height are equal one to the other.*

* *By height underſtand the perpendicular drawn from the plane of the baſe to the oppoſite plane.*

Firſt, let the parallelepipedons AB, CD have the ſides perpendicular to the baſes; and at the ſide CP being produced, [a] make the Pgr. PRTS equal and like to the pgr. KELA. [b] and ſo the parallelepipedon PRTSQVYX equal and like to the parallelepipedon AB. Produce OωE, NDſ, ωPZ, DQFF, ERB, ſVγ, TSZ, YXF; and draw Eſ, Bγ, ZF.

[a] 18. 6.
[b] 27, 11 & 10 *def.* 11.

The planes OωſN, CRVH, ZTYF. [c] are parallels one to the other; [d] and the Pgrs. ALEK, CDωO, PRTS, PRBZ, are equal. Therefore ſince the parallelepipedon CD. PVſω :: pgr. Cω (PRBZ) Pſ [e] :: parallelepipedon PRBZQVγF. PVſω; the parallelepipedon CD [f] ſhall be = PRBZQVγF [g] = PRVQSTYX [h] = AB. Which was to be demonſtrated.

[c] 30 *def.* 11
[d] *hyp. and* 35. 1.
[e] 25. 11.
[f] 9. 5.
[g] 29. 11.
[h] *conſtr.*

But if the parallelepipedons AB, CD, have ſides oblique to the baſe, then on the ſame baſes and in the ſame height, place parallelepipedons whoſe ſides are perpendicular to the baſe. [k] They ſhall be equal to one another, and thoſe that are oblique, [m] whence alſo the oblique parallelepipedons AB, CD, are equal. Which was to be demonſtrated.

[k] 29. 11.
[m] 1 *ax.* 1.

PROP. XXXII.

Solid parallelepipedons ABCD, EFGL, of the ſame height, are one to the other, as their baſes, AB, EF.

Produce EHI, [a] and make the pgr. FI = AB, and [b] compleat the parallepp. FINM. It is clear the parallepp. FINM. ([c] ABCD.) EFGL

[a] 45. 1
[b] 31. 1.
[c] 31. 1.

[d] 25. 11. EFGL [d] :: FI (AB). EF. Which was to be demonſtrated.

PROP. XXXIII.

Like ſolid parallelepipedons, ABCD, EFGH, are in tripled proportion one to the other, of that in which their homologous ſides or of like proportion AI, EK, are.

Produce the right lines AIL, DIO, BIN, and [a] make IL, IO, IN, equal to EK, KH, KF, [b] and ſo the parallepp. IXMT equal and like to the parallepp. EFGH. [c] let the parallepps. IXPB, DLYQ, be finiſhed. [d] Then ſhall be AI. IL (EK) :: DI. IO (HK) :: BI. IN. KF. [e] that is, the pgr. AD. DL :: DL. IX :: BO. IT. [f] *i. e.* the parallepp. ABCD. DLQY :: DLQY. IXBP :: IXBP. IXMT. ([g] EFGH) [h] therefore the proportion of ABCD to EFGH is triple of the proportion of ABCD to DLQY, [k] or of AI to EK. Which was to be demonſtrated.

[a] 3. 1.
[b] 27. 11.
[c] 31. 1.
[d] *hyp.*
[e] 1. 6.
[f] 32. 11,
[g] *conſtr.*
[h] 10 *def.* 5.
[k] 1. 6.

Corollarium.

Hence it appears, that if four right lines be continually proportional, as the firſt is to the fourth, ſo is a parallelepipedon deſcribed on the firſt to a parallelepipedon deſcribed on the ſecond, being like and in like manner deſcribed.

PROP. XXXIV.

In equal ſolid parallelepipedons ABCB, EHGF, the baſes and altitudes are reciprocal (AD. EH :: EG. AC.) And ſolid parallelepipedons, ADCB, EHGF, whoſe baſes and altitudes are reciprocal, are equal.

Firſt,

First, let the ſides CB, GE, be perpendicular to the baſes; then if the altitudes of the ſolids are equal, the baſes alſo ſhall be equal, and the thing is clear. But if the altitudes are unequal, from the greater EG [a] take EI = AC, and at I [b] draw the plane IK parallel to the baſe EH. then

1. *Hyp.* AD. EH [c] :: parallepp. ADCB. EHIK [d] :: parallepp. EHGF. EHIK [e] :: GL. IL [e] :: GE. IE. ([f] AC.) [g] it is plain therefore that AD. EH :: GE. AC. Which was to be demonſtrated.

2. *Hyp.* ADCB. EHIK [h] :: AD. EH [k] :: EG. EI [l] :: GL. IL [m] :: parallepp. EHGF. EHIK. [n] wherefore the parallelepipedon ADCB = EHGF. Which was to be demonſtrated.

Moreover, let the ſides be oblique to the baſes, and erect right parallelepipedons upon the ſame baſes in the ſame altitude; the oblique parallepps ſhall be equal to them. Wherefore ſince by the firſt part, the baſes and altitudes of thoſe be reciprocal, the baſes and altitudes of theſe alſo ſhall be reciprocal. Which was to be demonſtrated.

a 3. I.
b 31. I.
c 32. 11.
d 17. 5.
e 1. 6.
f *conſtr.*
g 11. 5.
h 32. 11.
k *hyp.*
l 1. 6.
m 32. 11.
n 9. 5.

Corollarium.

All that hath been demonſtrated of Parallelepipedons in the 29, 30, 31, 32, 33, 34. Prop. *does alſo agree to triangular priſms, which are half parallelepipedons, as appears by Prop.* 28. Therefore,

1. Triangular priſms are of equal height with their baſes.

2. If they have the ſame or equal baſes and the ſame altitude they are equal.

3. If they be like, their proportion is treble to that of their ſides of like proportion.

4. If they be equal, their baſes and altitudes are reciprocal; and if their baſes and altitudes be reciprocal, they are alſo equal.

PROP. XXXV.

If there be two plane angles BAC, EDF, equal, and from the points of thoſe angles two right lines AG, DH, be elevated on high, containing equal angles with the lines firſt given, each to his correſpondent angle (the angle GAB = HDE, and GAC = HDF.) and if in thoſe elevated lines AG, DH, ſome points be taken, G, H; and from theſe points perpendicular lines GI, HK, drawn to the planes BAC, EDF, in which the angles firſt given are, and right lines AI, DK, be drawn to the angles firſt given from the points IK, which are made by the perpendiculars in the planes; thoſe right lines with the elevated lines AG, DH, ſhall contain equal angles GAM, HDK.

Make DH, AL, equal; and GI, LM, parallels, and MC to AC, MB to AB, KF to DF, KE to DE perpendicular; and draw the right lines BC, LB, LC, and EF, HF, HE; [a] and LM is perpendicular to the plane BAC; [b] wherefore the Angles LMC, LMA, LMB; and by the ſame reaſon the angles HKF, HKD, HKE, are right angles. Therefore ALq [c] = LMq + AMq [c] = LMq + CMq + ACq [c] = LCq + ACq. [d] therefore the angle ACL is a right angle. Again ALq [e] = LMq + MAq [e] = LMq + BMq + BAq [e] = BLq + BAq. [d] therefore the angle ABL is alſo a right angle. By the like inference the angles DFH, DEH

[a] 8. 11.
[b] 3 *def.* 11.
[c] 47. 1.
[d] 48. 1.
[e] 47. 1.

DEH are right angles; [f] therefore AB = DE, [f] and BL = EH, [f] and AC = DF, and CL = FH. [g] wherefore alſo BC = EF; [g] and the angle ABC = DEF, [g] and the angle ACB = DFE. [h] whence the other right angles CBM, BCM, are equal to the other FEK, EFK. [k] therefore CM = FK, [l] and ſo alſo AM = DK. therefore if from LAq [m] = HDq be taken away AMq = DKq; [n] there remains LMq = HKq. wherefore the triangles LAM, HDK are equilateral one to the other; [o] therefore the angle LAM = HDK. Which was to be demonſtrated.

[f] 26. 1.
[g] 4. 1.
[h] 1 *ax.* 1.
[k] 26. 1.
[l] 47. 1.
[m] *conſtr.*
[n] 47. 1. & 3. *ax.*
[o] 8. 1.

Corollarium.

Therefore, if there be two plane angles equal, from whoſe points equal right lines be elevated on high, containing equal angles with the lines firſt given, each to each; Perpendiculars drawn from the extream points of thoſe elevated lines to the planes of the angles firſt given, are equal one to the other; *viz.* LM = HK.

PROP. XXXVI.

If there be three right lines DE. DG, DF, proportional, the ſolid parallelepipedon DH made of them, is equal to the ſolid parallelepipedon IN made of the middle line DG (IL) which is alſo equilateral, and equiangular to the ſaid parallelepp. DH.

Becauſe DE. IK [a] :: IL. DF. [b] the pgr. LK ſhall be = FE. and by reaſon of the equality of the plane angles at E and I, and of the lines GD, IM, alſo the altitudes of the parallelepipedons are equal by the prec. Coroll. [c] there-

[a] *hyp.*
[b] 14. 6.
[c] 31. 11.

fore the parallepps are equal one to the other. Which was to be demonstrated.

PROP. XXXVII.

If there be four right lines A, B, C, D, proportional, the solid parallelepipedons A, B, C, D, being like, and in like sort described from them, shall be proportional. And if the solid parallelepipedons, being like and in like sort described, be proportional, (A. B :: C. D.) then those right lines A, B, C, D, shall be proportional.

For the proportions of the parallelepps. [a] are triple of those of the lines; therefore if A. B :: C. D. [b] then shall the parallepp. A. parallepp. B :: parallepp. C. parallepp. D. and so also contrarily.

[a] 33. 11.
[b] *Sch.* 23. 5.

PROP. XXXVIII.

If a plane AB be perpendicular to a plane AC, and a perpendicular line EF be drawn from a point E in one of the planes (AB) to the other plane AC, that perpendicular EF shall fall upon the common section of the planes AD.

If it be possible, let F fall without the intersection AD. and in the plane AC [a] draw FG perpendicular to AD; and joyn EG. The angle FGE [b] is a right angle, and EFG is supposed to be such also; therefore two right angles are in the triangle EFG. [c] Which is absurd.

[a] 12. 1.
[b] 4. *and* 3. *def.* 11.
[c] 17. 1.

PROP. XXXIX.

If the sides (AE, FC, AF, EC, and DH, GB, DG, HB) of the opposite planes AC, DB, of a solid parallelepipedon AB, be divided into two equal parts, and

and planes ILQO, PKMR, be drawn through their ſections, the common ſection of the planes ST, and the diameter of the ſolid parallelepipedon AB ſhall divide one the other into two equal parts.

Draw the right lines SA, SC, TD, TB. Becauſe [a] the ſides DO, OT are equal to the ſides BQ, QT, [b] and the alternate angles TOD, TQB equal alſo, [c] the baſes DT, TB, and the angles DTO, BTQ, are equal. [d] therefore DTB is a right line, and ſo in like manner is ASC. Moreover [e] as well AD is parallel and equal to FG [e] as FG to CB, and [f] thence AD is parallel and equal to CB; [g] and conſequently AC to DB. [h] wherefore AB and ST are in the ſame plane ABCD. Therefore ſince the vertical angles AVS, BVT, and the alternate angles ASV, BTV are equal; [k] and AS = BT; therefore ſhall AV be = BV, [l] and SV = VT. Which was to be demonſtrated.

[a] 34. 1.
[b] 29. 1.
[c] 4. 1.
[d] *ſch.* 15. 1.
[e] 34. 1.
[f] 9. 11 & 1 *ax.*
[g] 33. 1.
[h] 7. 11.
[k] 7 *ax.* 1.
[l] 26. 1.

Corollarium.

Hence, in every parallelepipedon all the diameters biſect one another in one point, V.

PROP. XL.

If two Priſms ABCFED, GHMLIK, be of equal altitude, whereof one hath its baſe ABCF a parallelogram, and the other GHM a triangle; and if the parallelogram ABCF be double to the triangle GHM; thoſe priſms ABCFED, GHMLIK are equal.

For if the parallepps. AN, GQ, be compleated, [a] they ſhall be equal, becauſe of the equality [b] of the Baſes AC, GP, and [c] of the altitudes. [d] therefore alſo the priſms, [e] the

[a] 31. 11.
[b] 34. 1. & 7. *ax.*
[c] *hyp.*
[d] 28. 11.
[e] 7. *ax.* 1

halfs thereof, ſhall be equal. Which was to be demonſtrated.

Schol.

From the preceding Demonſtrations, the Dimenſion of triangular priſms, and quadrangular, or parallelepipedons, is learnt; viz. *by multiplying the altitude into the baſe.*

Andr. Tacq.

As if the altitude be 10 foot, and the baſe 100 ſquare foot (the baſe may be meaſured by *ſch.* 35. 1. or by 41. 1.) then multiply 100 by 10. and 1000 cubic foot ſhall be produced for the ſolidity of the priſm given.

For as a rectangle, ſo alſo is a right parallelepipedon produced of the altitude multiplied into the baſe. Therefore every parallelepipedon is produced of the altitude multiplied into the baſe, as appears by 31. of this Book.

Moreover, ſince the whole Parallelepipedon is produced of the altitude drawn into the baſe, the half thereof (that is, a triangular priſm) ſhall be produced of the altitude drawn into half the baſe, namely the Triangle.

An Advertiſement.

Obſ. That of thoſe Letters which denote a ſolid angle, the firſt is always at the point in which the angle is; but of thoſe Letters which denote a pyramid, the laſt is at the ſupreme point thereof.

Ex. gr. the ſolid angle ABCD is at the point A; and the ſupreme point of the Pyramid BCDA is at the point A. and the baſe is the Triangle BCD.

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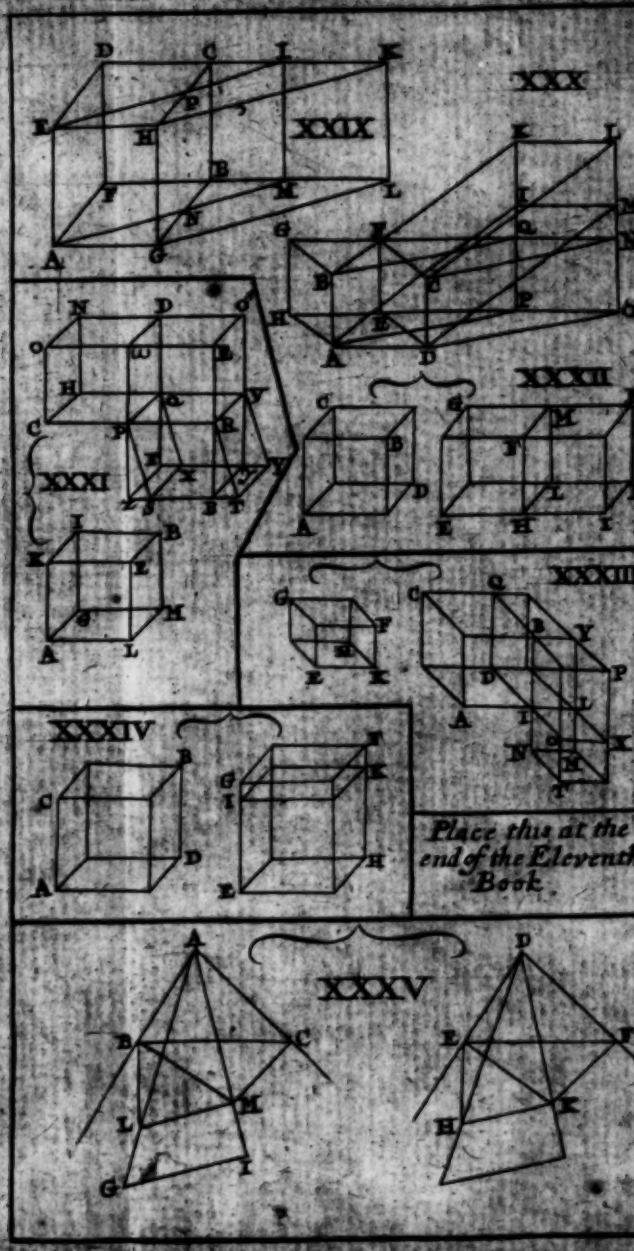
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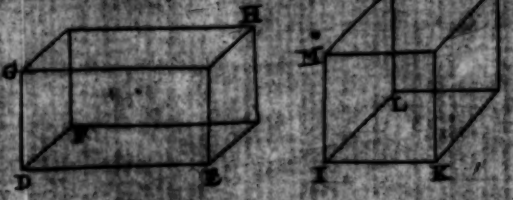

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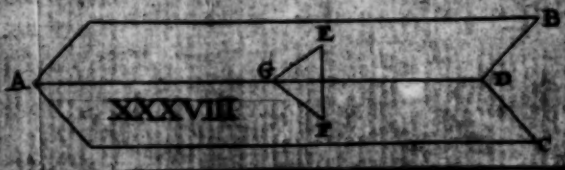

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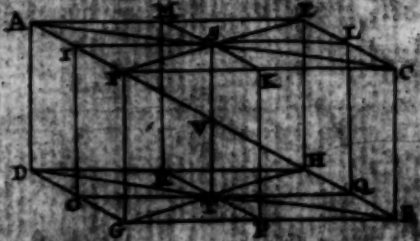


XL

THE TWELFTH BOOK OF Euclid's *Elements*.

PROP. I.

Like polygonous figures ABCDE, FGHIK, described in circles ABD, FGI, are one to another, as the squares described of the diameters of the circles AL, FM.

Draw AC, BL, FH, GM. Because [a] the angle ABC = FGH, [a] and AB. BC :: FG. GH. [b] therefore shall the angle ACB ([c] ALB) be = FHG ([c] FMG.) but the angles ABL, FGM [d] are right and so equal; [e] therefore the triangles ABL, FGM, are equiangular. [f] wherefore AB. FG :: AL. FM. [g] therefore ABCDE. FGHIK :: ALq. FMq.

[a] 1 *def.* 6.
[b] 6. 6.
[c] 21. 3.
[d] 31. 3.
[e] 32. 3.
[f] *cor.* 4. 6.
[g] 22. 6.

Corollarium.

Hence (because AB. FG :: AL. FM :: BC. GH, *&c.*) the contents of like polygonous figures

[h] 1.12. *and* 12.5. gures deſcribed in a circle are in [h] proportion as the diameters.

PROP. II.

Circles ABT, EFN, are in proportion one to another, as the ſquares of their diameters AG, EG are.

Suppoſe ACq. EGq :: the circle ABT. I. I ſay then I is equal to the circle EFN.

For firſt, if it be poſſible, let I be leſs than the circle EFN, and let K be the exceſs or difference. Inſcribe the ſquare EFGH in the [a] *ſch.* 7. 4. circle EFN, [a] it being the half of a circumſcribed ſquare, and ſo greater than the ſemicircle. [b] 30. 3. [b] Divide equally in two the Arches EF, FG, GH, HE, and at the points of the diviſions joyn the right lines EL, LF, &c. at L draw [c] *ſch.* 27. 3. the Tangent PQ ([c] which is parallel to EF) and produce HEP, GFQ. then is the triangle [d] 41. 1. ELF [d] the half of the pgr. EPQF, and ſo greater than the half of the ſegment ELF; and in like ſort, the reſt of thoſe Triangles exceed the halfs of the reſt of the ſegments. And if the arches EL, LF, FM, *&c.* be again biſected, and the right lines joyned, the Triangles will likewiſe exceed the half of the ſegments. Wherefore if the ſquare EFGH be taken from the circle EFN, and the triangles from the other ſegments, and this be done [e] 1, 10. continually, at length [e] there will remain ſome magnitude leſs than K. Let us have gone ſo far, namely to the ſegments EL, LF, FM, *&c.* taken together leſs than K. Therefore I ([f] the [f] *hyp. and* 3 *ax.* circle EFN—K) = the polyg. ELFMNHO [g] 30. 3. *and* 2 *poſt.* 1. (the circle EFN— the ſegm. EL + LF, &c.) In the circle ABT [g] conceive a like polygonon

non AKBSCTDV inſcribed. therefore ſince AKBSCTDV . ELFMGNHO [h] :: ACq. EGq [k] :: the circle ABT . I. and the polygonon AKBSCTDV [l] ▭ the circle ABT. the polyg. ELFMGNHO [m] ſhall be ▭ I. but before, I was ▭ ELFMGNHO. Which is repugnant.

[h] I. 12.
[k] *hyp.*
[l] 9. *ax.* 1.
[m] 14. 5.

Again, if it be poſſible, let I be ▭ the circle EFN. Therefore becauſe ACq. EGq [a] :: the circle ABT . I; and inverſely I. the circle ABT :: EGq . ACq. ſuppoſe I. the circle ABT :: the circle EFN. K. [o] therefore the circle ABT ▭ K. [p] and EGq . ACq :: the circle EFN . K. Which is ſhewn to be repugnant.

[a] *hyp.*
[o] 14. 5.
[p] 11. 5.

Therefore it muſt be concluded, that I is = to the circle EFN. Which was to be demonſtrated.

Corollarium.

Hence it follows, That as a circle is to a circle, ſo is a polygonon deſcribed in one to a like polygonon deſcribed in the other.

PROP. III.

Every Pyramid ABDC having a triangular baſe, may be divided into two Pyramids AEGH, HIKC, equal, and like one to the other, having baſes triangular, and like to the whole ABDC; and into two equal Priſms, BFGEIH, EGDIHK; which two Priſms are greater than the half of the whole Pyramid ABDC.

Divide the ſides of the pyramid into two parts at the points E, F, G, H, I, K. and joyn the right lines EF, FG, GE, EI, IF, FK, KG, GH, HE. Becauſe the ſides of the pyramid are

pro-

proportionally cut, [a] thence HI, AB; and GF, AB; and IF, DC; and HG, DC, &c. are parallels. and consequently HI, FG; and GH, FI are also parallels. therefore it is apparent, that the triangles ABD, AEG, EBF, FDG, HIK, [b] are equiangular. and that the four last are [c] equal: in like manner the triangles AGB, AHE, EIB, HIC, FGK, are equiangular; and the four last are equal one to the other. Also the triangles BFI, FDK, IKC, EGH; and lastly the triangles AHG, GDK, HKC, EFI, are like and equal. Moreover, the triangles HIK to ADB, and EGH to BDC, and EFI to ADC, and FGK to ABC, [d] are parallel. From whence it evidently follows, first, that the pyramids AEGH, HIKC, are equal, and [e] like to the whole ABDC, and to one another. Next, that the solids BFGEIH, FGDIHK, are prisms, and that of equal height, as being placed between the parallel planes ABD, HIK. but the base BFGE is [f] double of the base FDG. wherefore the said prisms are [g] equal; whereof the one BFGEIH, is greater than the pyramid BEFI, that is, than AEGH, the whole than its part; and consequently the two prisms are greater than the two pyramids, and so exceed the half of the whole pyramid ABDC. Which was to be demonstrated.

[a] 2. 6.
[b] 29. 1.
[c] 26. 1.
[d] 15. 11.
[e] 10 *def.* 11
[f] 2 *ax.* 1.
[g] 40. 11.

PROP. IV.

If there be two pyramids ABCD, EFGH, of the same altitude, having triangular bases ABC, EFG; and either of them be divided into two pyramids (AILM, MNOD; and EPRS, STVH) equal one to the other and like to the whole, and into

into two equal prisms (IBKLMN, KLCNMO; and PFQRST, QRGTSV;) and if in like manner either of those pyramids made by the former division be divided, and this be done continually; then as the base of one pyramid is to the base of the other pyramid, so are all the prisms which are in one pyramid, to all the prisms which are in the other pyramid, being equal in multitude.

For (applying the construction of the precedent *Prop.*) BC. KC [a] :: FG. QG. [b] therefore the triangle ABC is to the like triangle LKC as EFG is to [c] the like RQG. therefore by permutation ABC. EFG [d] :: LKC. RQG [e] :: the prism KLCNMO. QRGTSV (for these are of equal altitude) [f] :: IBKLMN. PFQRST. [g] wherefore the triangle ABC. EFG :: the prism KLCMNO + IBKLMN. the prism QRGTSV + PFQRST. Which was to be demonstrated.

[a] 15. 5.
[b] 22 6.
[c] 2. 6. &c.
[d] 16. 5.
[e] *sch.* 34. 11
[f] 7. 5.
[g] 12. 5.

But if the pyramids MNOD, AILM; and EPRS, STVH; be further divided, in like manner the four new prisms made hereby shall be to the four produced before as the bases MNO and AIL are to the bases STV, and EPR; that is, as LKC to RQG, or as ABC to EFG. [h] wherefore all the prisms of the pyramid ABCD are to all the prisms of the pyramid EFGH as the base ABC is to the base EFG. Which was to be demonstrated.

[h] 12. 5.

PROP. V.

Pyramids ABCD, EFGH, being under the same altitude, having triangular bases ABC, EFG, are one to another as their bases ABC, EFG, are.

Let

Let the triangle ABC. EFG :: ABCD. X. I ſay X is equal to the pyramid EFGH. For if it be poſſible, let X be ▱ EFGH, and let the exceſs be Y. Divide the pyramid EFGH into priſms and pyramids, and the other pyramids in like manner, [a] till the pyramids left EPRS, STVH, be leſs than the ſolid Y. Therefore ſince the pyramid EFGH = X + Y, it is manifeſt, that the remaining priſms PFQRST, QRGTSV, are greater than the ſolid X. Conceive the pyramid ABCD divided after the ſame manner, [b] then will be the priſm IBKLMN + KLCNMO. PFQRST + QRGTSV :: ABC. EFG [c] :: the pyr. ABCD. X. [d] therefore X ▱ the priſm PFQRST + QRGTSV; which is contrary to that which was affirmed before.

[a] 11. 10.
[b] 4. 12.
[c] hyp.
[d] 14. 5.

Again, conceive X ▱ the pyr. EFGH, and make the pyr. EFGH. Y :: X. the pyr. ABCD [e] :: EFG. ABC. Becauſe EFGH [f] ▱ X, [g] thence Y ▱ the pyr. ABCD. which is ſhewn before to be impoſſible. Therefore I conclude, that X is equal to the pyr. EFGH. Which was to be demonſtrated.

[e] hyp. and cor. 4. 5.
[f] ſuppoſ.
[g] 14. 5.

PROP. VI.

Pyramids ABCDEF, GHIKLM, conſiſting under the ſame altitude, and having polygonous baſes ABCDE, GHIKL, are to one another as their baſes ABCDE, GHIKL are.

Draw the right lines AC, AD, GI, GK. then is the baſe ABC. ACD [a] :: the pyr. ABCF. ACDF. [b] therefore by compoſition, ABCD. ACD :: the pyr. ABCDF. ACDF. [c] but alſo ACD. ADE :: the pyr. ACDF. ADEF. [d] therefore of equality ABCD. ADE :: ABCDF. ADEF. and [e] thence by compoſition ABCDE. ADE :: the

[a] 5. 12.
[b] 18. 5.
[c] 22. 5.

the pyr. ABCDEF. ADEF. moreover ADE. GKL [d] :: the pyr. ADEF. GKLM; and, as before, and inversely GKL. GHIKL :: the pyr. GKLM. GHIKLM. [e] therefore again of equality ABCDE. GHIKL :: the pyr. ABCDEF. GHIKLM. Which was to be demonstrated. [d] 5. 12.

If the bases have not sides of equal multitude, the demonstration will proceed thus. The base ABC. GHI [e] :: the pyr. ABCF. GHIK. [e] and ACD. GHI :: the pyr. ACDF. GHIK. [f] therefore the base ABCD. GHI :: the pyr. ABCDF. GHIK. [e] Moreover the base ADE. GHI :: the pyr. ADEF. GHIK. [f] therefore the base ABCDE. GHI :: the pyr. ABCDEF. GHIK. [e] 5. 12. [f] 24. 5.

PROP. VII.

Every prism, ABCDEF, having a triangular base, may be divided into three pyramids ACBF, ACDF, CDFE, equal one to the other, and having triangular bases.

Draw the diameters of the parallelograms, AC, CF, FD. Then the triangle ACB is [a] = ACD. [b] therefore the pyramids of equal height ACBF, ACDF, are equal. In like manner, the pyr. DFAC = the pyr. DFEC. but ACDF and DFAC are one and the same pyramid. [c] therefore the three pyramids ACBF, ACDF, DFEC, into which the prism is divided, are equal one to the other, Which was to be demonstrated. [a] 34. 1. [b] 5. 12. [c] 1. *ax.* 1.

Hence, every pyramid is the third part of the prism that has the same base and height with it; or every prism is treble of the pyramid that has the same base and height with it.

For

For reſolve the polygonous priſm ABCDEGHIKF into triangular priſms; and the pyramid ABCDEH into triangular pyramids;
[a] 7. 12. [a] then all the parts of the priſm ſhall be treble to all the parts of the Pyramid, [b] conſequently the whole priſm ABCDEGHIKF is
[b] 1. 5. treble to the whole pyramid ABCDEH. Which was to be demonſtrated.

PROP. VIII.

Like pyramids ABCD, EFGH, which have triangular baſes ABC, EFG are in triple proportion of that in which their ſides of like proportion AC, EG, are.

[a] 27. 11. [a] Compleat the parallelepipedons ABICDMKL, EFNGHQOP, which [b] are like, and
[b] 9 *def.* 11. [c] 28. 11. & 7. 12. [c] ſextuple of the pyramids ABCD, EFGH. [d] and therefore in the ſame proportion with
[d] 15. 5. them one to another, [e] that is, triple of that
[e] 33. 11. of the ſides of like proportion, *&c.*

Corollarium.

Hence, alſo like polygonous pyramids have proportion tripled to that of the ſides of like proportion; as may eaſily be proved by reſolving the ſame into triangular pyramids.

PROP. IX.

See the precedent Scheme.

In equal pyramids ABCD, EFGH, having triangular baſes ABC, EFG, the baſes and altitudes are reciprocal; And pyramids having triangular baſes, whoſe altitudes and baſes are reciprocal, are equal.

1. *Hyp.*

1. *Hyp.* The compleated parallelepipedons ABICDMKL, EFNGHQOP, are [a] ſextuple of the equal pyramids ABCD, EFGH (either of either) and ſo equal one to the other. therefore the altitude (H.) the alt. (D) [b] :: ABIC. EFNG [c] :: ABC. EFG. Which was to be demonſtrated.

[a] 28. 11. *and* 7. 12.
[b] 34. 11.
[c] 15. 5.

2. *Hyp.* The altitude (H.) the alt. (D) [d] :: ABC. EFG [e] :: ABIC. EFNG. [f] therefore the parallelepipedons ABICDMKL, EFNGHQOP are equal. [g] conſequently alſo the pyramids ABCD, EFGH being ſubſextuple of the ſame, are equal. Which was to be demonſtrated.

[d] *hyp.*
[e] 15. 5.
[f] 34. 11.
[g] 6. *ax.* 1.

The ſame is applicable to polygonous Pyramids; for they may alſo in like manner be reduced to triangulars.

Corollarium.

Whatſoever is demonſtrated of pyramids in prop. 6, 8, 9, *does likewiſe agree to any ſort of priſms; ſeeing they are triple of the pyramids that have the ſame baſe and altitude with them.* Therefore,

1. The proportion of priſms of equal altitude is the ſame with that of their baſes.

2. The proportion of like priſms is triple of that of the ſides of like proportion.

3. Equal priſms have their baſes and altitudes reciprocal; and priſms which are ſo reciprocal, are equal.

Scholium.

From what is hitherto demonſtrated, the dimenſion of any priſms and pyramids may be collected.

[a] The

[a] 1 *cor.* 12. *and* *sch.* 40. 11. [b] 7. 12.

[a] The ſolidity of a priſm is produced of the altitude multiplied into the baſe; [b] and therefore likewiſe that of a pyramid, of the third part of the altitude multiplied into the baſe.

PROP. X.

Every Cone is the third part of a Cylinder having the ſame baſe with it ABCD, and the altitude equal.

See the ſecond fig. of this Book.

If you deny it, then firſt let ſuch cylinder be more than triple to the Cone, and let the exceſs be E. A priſm deſcribed on a ſquare in the circle ABCD [a] is ſubduple of a priſm deſcribed upon a ſquare about the circle, being equal to it and the cylinder in height. Therefore a priſm upon the ſquare ABCD exceeds the half of the cylinder. And likewiſe a priſm upon the baſe AFB, of equal height to the cylinder, [b] is greater than the half of the ſegment of the cylinder AFB. continue an equal biſection of the Arches, and ſubtract the priſms till the remaining ſegments of the Cylinder, namely at AF, FB, *&c.* become leſs than the ſolid E. Therefore the cyl. — ſegm. AF, FB, *&c.* (the priſm on the baſe AFBGCHDI) [c] is greater than the cylinder — E ([d] the triple of the cone.) therefore the pyramid, [e] a third part of the ſaid Priſm (being placed on the ſame baſe, and of the ſame height) is greater than the cone of equal height on the baſe ABCD a circle, *i.e.* the part greater than the whole. Which is abſurd.

[a] *ſch.* 7. 4. & *cor.* 9. 12.

[b] *ſch.* 27. 3. *and cor.* 9. 12.

[c] 5. *ax.* 1. [d] *hyp.* [e] *cor.* 7. 12.

But if the Cone be affirmed to be greater than the third part of the Cylinder, then let the Exceſs be E. Detract the pyramids from the cone, as you did in the firſt part the priſms from

from the cylinder, till ſome ſegments of the Cone remain, conceive at AF, FB, BG, &c. leſs than the ſolid E. therefore the Cone — E (f ⅔ of the cylin.) ⊐ the pyr. AFBGCHDI (the cone — ſegm. AF, FB, &c.) therefore the priſm triple to the pyramid (*viz.* of equal height, and on the ſame baſe) is greater than the cylinder on the baſe ABCD, the part than the whole. Which is abſurd. Wherefore it muſt be granted, that the cylinder is equal to the triple of the cone. Which was to be demonſtrated. f *hyp.*

PROP. XI.

Cylinders and Cones ABCDK, EFGHM, being under the ſame altitude, are to one another as their baſes ABCD, EFGH, are.

Let the circle ABCD. the cir. EFGH :: the cone ABCDK. N. I ſay, N is equal to the cone EFGHM.

For if it be poſſible, let N be ⊐ the cone EFGHM, and let the Exceſs be O. The preparation and argumentation of the prec. Prop. being ſuppoſed; then ſhall O be greater than the ſegments of the cone EP, PF, FQ, &c. and ſo the ſolid N ⊐ the pyr. EPFQGRHSM. In the circle ABCD [a] make a like polygonous fig. ATBVCXDY. Becauſe the pyr. ABVYK. the pyr. EFQSM [b] :: the polyg. ATBVY. the polyg. EPFQS [c] :: the cir. ABCD. the cir. EFGH [d] :: the cone ABCDK. N. [e] thence the pyr. EPFQGRHSM ſhall be ⊐ N. contrary to what was affirmed before. Again conceive N ⊏ the cone EFGHM. and make the cone EFGHM. O :: N. the cone ABCDK [f] :: the circ. EFGH. ABCD. [g] therefore O ⊐ the cone ABCDK; a 30. 3. & 1. *poſt.* b 6. 12. c *cor.* 2. 12. d *hyp.* e 14. 5. f *hyp.* & *by inverſion.* g 14. 5.

ABCDK; which is abſurd, as appears by what is ſhewn in the firſt part.

Therefore rather admit ABCD . EFGH:: the cone ABCDK . EFGHM. Which was to be demonſtrated.

The ſame may be demonſtrated of cylinders, if cylinders and priſms be conceived in the place of cones and pyramids. therefore, &c.

Scholium.

Hence, is gathered the dimenſion of all ſorts of Cylinders and Cones. The ſolidity of a right cylinder is produced of the circular baſe ([a] the dimenſion whereof is to be learnt out of *Archimedes*) multiplied into the height; [b] whence in like manner that of every cylind.

[a] 1 *Prop. de dimenſ. circ.*

[b] 11. 12.

Therefore the ſolidity of a cone is produced of the third part of the altitude multiplied into the baſe.

PROP. XII.

Like Cones and Cylinders ABCDK, EFGHM, are in triple proportion of that of the diameters TX, PR, of their baſes ABCD, EFGH.

Let the cone A have to N triple proportion of TX to PR. I ſay, N is = the cone EFGHM. For if it be poſſible, let N be ⊐ EFGHM. and let the exceſs be O. therefore N ⊐ the pyr. EPFQGRHSM. Let the axes of the cones be IK, LM, and joyn the right lines VK, CK, VI, CI, and QM, GM, QL, GL.

[a] 14 *def.* 11 Becauſe the cones are like, [a] thence VI. IK ::

[b] 18 *def.* 11 QL. LM. but the angles VIK, QLM, [b] are

[c] 6. 6. right angles. [c] therefore the triangles VIK.

[d] 4. 6. QLM, are equiangul. [d] whence VC. VI :: QG.

QG. QL. also VI. VK :: QL. QM. therefore of equality VC. VK :: QG. QM. [c] moreover VK. CK :: QM. MG. therefore again of equality VC. CK :: QG. GM. [f] therefore the triangles VKC, QMG, are like: and by the same reason the other triangles of this pyramid are like to the other of that. [g] wherefore the pyramids themselves are like. [h] But they are in triple proportion of that of VC to QG. [k] that is, of VI to RL, [l] or TX to PR. [m] therefore the pyr. AIBVCXDYK. the pyr. EPFQGRHSM :: the cone ABCDK. N. [n] whence the pyr. EPFQGRHSM ⊐ N. which is repugnant to what was affirmed before.

[c] 7. 5.
[f] 5. 6.
[g] 9 *def.* 11.
[h] *cor.* 8. 12
[k] 4. 6.
[l] 15. 5.
[m] *hyp. and* 11. 5.
[n] 14. 5.

Again, take N ⊏ the cone EFGHM. make the cone EFGHM. O :: N. the cone ABCDK [o] :: the pyr. EPRM. ATCK [p] :: GQ. VC thrice :: [q] PR. TX thrice. but O [r] is ⊐ ABCDK. which was before shewn to be repugnant. Wherefore N = the cone EFGHM. Which was to be demonstrated.

[o] *before, & inversely.*
[p] *cor.* 8. 12.
[q] 4. 6.
[r] 14. 5.

But forasmuch as what proportion soever cones have, also Cylinders, being triple of them, have the same; therefore cylinder to cylinder shall have proportion triple of the diameters of the bases.

PROP. XIII.

If a Cylinder ABCD be divided by a plane EF parallel to the opposite planes BC, AD, then as one cylinder AEFD is to the other cylinder EBCF, so is the Axis GI to the Axis IH.

The Axis being produced, [a] take GK = GI, and HL = IH = LM. and conceive planes drawn at the points K, L, M. parallel to the circles AD, BC. [b] therefore the cylinder FD = the

[a] 3. 1.
[b] 11. 12.

[b] 11. 12. = the cyl. AN. and the cyl. EC [b] = BO [b] = OP. therefore the cylinder EN is as multiplex of the cylinder ED as the axis IK is of the axis IG. and in like manner the cylinder FP is as multiplex of the cylinder EF, as the axis IM is of the axis IH. but as IK is =, ⊏, ⌝ IM, [c] so is the cylinder EN =, ⊏, ⌝ FP. [d] therefore the cylinder AEFD. the cyl. EBCF :: GI. IH. Which was to be demonstrated.

[c] 11. 12.
[d] 6. *def.* 5.

PROP. XIV.

Cones AEB, CFD, and cylinders AH, CK, consisting upon equal bases AB, CD, are to one another as their altitudes ME, NF, are.

The cylinder HA, and the axis EM being produced, take ML = FN; and at the point L draw a plane parallel to the base AB, [a] then shall the cyl. AP be = CK. [b] but the cyl. AH. AP (CK) :: ME. ML (NF.) Which was to be demonstrated.

[a] 11. 12.
[b] 13. 12.

The same may be affirmed of cones subtriple of cylinders; * as also of prisms and pyramids.

* *apply* 9, *and* 7. 12.

PROP. XV.

In equal cones BAC, EDF, and cylinders BH, EK, the bases and altitudes are reciprocal (BC. EF :: MD. LA:) And cones and cylinders, whose bases and altitudes are reciprocal, are equal one to the other.

If the altitudes be equal then the bases are equal too, and the thing is evident. If unequal, then take away MO = LA.

1. *Hyp.*

1. *Hyp.* Then is MD.MO ([a] LA) [b] :: the cyl.EK. ([c] BH) EQ [d] :: the cir. BC.EF. Which was to be demonſtrated.

2. *Hyp.* BC.EF [e] :: DM.OM (LA) [f] :: the cyl. EK.EQ [g] :: BC.EF [h] :: BH.EQ. [k] Therefore the cylind. EK = BH. Which was to be demonſtrated.

The ſame Argument may be uſed for Cones.

[a] 14. 12.
[b] *conſtr.*
[c] *hyp.*
[d] 11. 12.
[e] *hyp.*
[f] 11. 12.
[g] 11. 5.
[h] 11. 12.
[k] 9. 5.

PROP. XVI.

Two unequal circles ABCG, DEF, having the ſame center M, to inſcribe in the greater circle ABCG a polygonous figure of equal and even ſides, which ſhall not touch the leſſer circle DEF.

Through the center M draw the line AC cutting the circle DEF in F, from whence raiſe a perpendicular FH. [a] divide the ſemicircle ABC into two equal parts; and the half thereof BC alſo; and ſo do continually. [b] till the arch IC become leſs than the arch HC. from I let fall the perpendicular IL. It is manifeſt, that the arch IC meaſures the whole circle, and that the number of Arches is even, and ſo that the ſubtended line IC is the ſide [c] of the polygonon that may be inſcribed without touching the leſſer circle DEF. For HG [d] touches the circle DEF, [e] to which IK is parallel, and placed outwardly; [f] wherefore IK does not touch the circle DEF; much leſs do CI, CK, and the other ſides of the polygonon more remote from the center. Which was to be done.

Coroll. Obſerve that IK touches not the circle DEF.

[a] 30. 3.
[b] 1. 10.
[c] *ſcb.* 16. 4.
[d] *cor.* 16. 3.
[e] 28. 1.
[f] 34. *def.* 1

PROP. XVII.

Two ſpheres ABCV, EFGH, conſiſting about the ſame center D, being given; to inſcribe a ſolid of many ſides (or Polyedron) in the greater ſphere ABCV, which ſhall not touch the ſuperficies of the leſſer ſphere EFGH.

Let both the ſpheres be cut by a plane paſſing by the center making the circles EFGH, ABCV; and the diameters AC, BV drawn, cutting perpendicularly. In the circle ABCV [a] inſcribe the equilateral polygone VMLNC, &c. not touching the circle EFGH: then draw the diameter Nα, and erect DO perpendicular to the plane ABC; by DO, and by the diameters AC, Nα, conceive planes DOC, DON erected, which ſhall be [b] perpendicular to the circle ABCV, and ſo in the ſuperficies of the ſphere make [c] the quadrants DOC, DON. In which let the right lines CP, PQ, QR, RO, NS, ST, Tγ, γO [d] be fitted, equal, and of equal multitude with CN, NL, &c. make the ſame conſtruction in the other quadrants OL, OM, &c. and in the whole ſphere. Then I ſay the thing required is done.

From the points P, S, to the plane ABCV, draw the perpendiculars PX, SY, [e] which ſhall fall on the ſections AC, Nα. Therefore becauſe both [f] the right angles PXC, SYN, [g] and PCX, SNY inſiſting on [h] equal circumferences, [f] are equal, the triangles alſo PCX, SNY [h] are equiangular. Wherefore being PC [k] = SN, [l] alſo is PX = SY, [l] and XC = YN; [m] whence DX = DY, [n] and therefore DX.XC :: DY.YN. [o] therefore YX, NC, are parallels. but becauſe PX, SY are equal,

and

a 16. 12.
b 18. 11.
c *cor.* 33. 6.
d 4. 1.
e 38. 11.
f 12. *ax.*
g 27. 3.
h 32. 1.
k *conſtr.*
l 26. 1.
m 3. *ax.* 1.
n 7. 5.
o 2. 6.

and ſince being perpendicular to the ſame plane ABCV, they are alſo [p] parallels, [q] therefore YX, SP ſhall be equal and parallels. [r] whence SP, NC, are parallel one to the other; and ſo the [ſ] quadrilateral NCPS, and by the ſame reaſon SPQT, TQRG, as alſo the [t] triangle γRO are ſo many planes. In like manner the whole ſphere may be ſhewn full of ſuch quadrilaterals and triangles; wherefore the figure inſcribed is a polyedron.

From the center D [u] draw DZ perpendicular to the plane NCPS; and joyn ZN, ZC, ZS, ZP. Becauſe DN. NC [x] :: DY. YX. thence NC is [y] ⊏ YX (SP,) and likewiſe SP ⊏ TQ, and TQ ⊏ γR. And becauſe the angles DZC, DZN, DZS, DZP, [z] are right, and the ſides DC, DN, DS, DP, [a] equal, and DZ common, [b] thence ZC, ZN, ZS, ZP are equal one to the other; and conſequently about the quadrilateral NCPS [c] a circle may be deſcribed, in which (becauſe NS, NC, CP, are [d] equal, and NC ⊏ SP) NC [e] ſubtends more than the quadrant, [f] therefore the ang. NZC at the center is obtuſe. [g] therefore NCq ⊏ 2 ZCq (ZCq + ZNq.) Let NI be drawn perpendicular to AC. therefore ſince the angle ADN ([h] DNC + DCN) [k] is obtuſe, the half of it DCN ſhall be greater than the half of a right angle; and ſo that which remains of the right ang. CNI ſhall be leſs than it, [n] whence IN ⊏ IC. therefore NCq (NIq + ICq) [o] ⊐ 2 INq. therefore IN ⊏ ZC. and conſequently DZ [p] ⊏ DI. but the point I is [q] without the ſphere EFGH. and ſo much more the point Z. wherefore the plane NCPS, (whoſe [r] next point to the center is Z,) does not touch the ſphere EFGH. And if a perpendicular Dδ be

[p] 6. 11.
[q] 33. 1.
[r] 9. 12.
[ſ] 7. 11.
[t] 2. 11.
[u] 11. 11.
[x] 4. 6.
[y] 14. 5.
[z] 3 *def.* 11
[a] 15. *def.* 11
[b] 47. 1.
[c] 15 *def.* 1
[d] *conſtr.*
[e] 28. 3.
[f] 33. 9.
[g] 12. 2.
[h] 32. 1.
[k] 9. *ax.* 1.
[l] 5. 1.
[n] 19. 1.
[o] 47. 1.
[p] 47. 1.
[q] *cor.* 19. 12.
[r] 47. 1.

drawn to the plane SPQT, the point *s*, and so also the plane SPQT is yet farther removed from the center; which is also true of the other planes of the polyedron. Therefore the polyedron ORQPCN, *&c.* inscribed in the greater sphere, does not touch the lesser. Which was to be done.

Corollarium.

Hence it follows, *that if in any other sphere a solid polyedron, like to the aboveſaid solid polyedron, be inſcribed, the proportion of the polyedron in one ſphere to the polyedron in the other is triple of that of the diameters of the Spheres.*

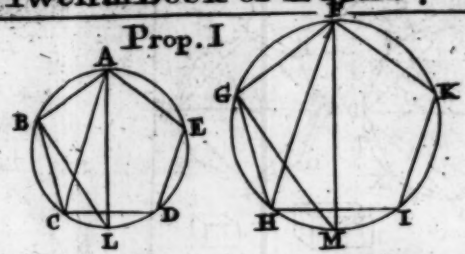
For if right lines be drawn from the centers of the ſpheres to all the angles of the baſes of the ſaid polyedrons, then the polyedrons will be divided into pyramids equal in number and like; whoſe homologous ſides are ſemidiameters of the ſpheres; as appears, if the leſſer of theſe ſpheres be conceived deſcribed within the greater about the ſame center. For the right lines drawn from the center of the ſphere to the angles of the baſes, will agree one to the other by reaſon of the likeneſs of the baſes; and ſo will like pyramids be made. Wherefore ſince every pyramid in one ſphere to every pyramid like it in the other ſphere, [a] has proportion triple to that of the homologous ſides, that is, of the ſemidiameters of the ſpheres; and [b] as one pyramid is to one pyramid, ſo all the pyramids, that is, the ſolid polyedron compoſed of theſe, are to all the pyramids, that is, the ſolid polyedron compoſed of the others; therefore the polyedron of one ſphere ſhall have to the poly-

[a] *cor.* 8. 12.

[b] 12. 5.

The Twelfth Book of Euclid.

Prop. I



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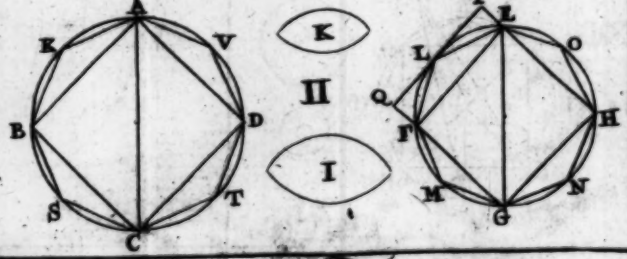

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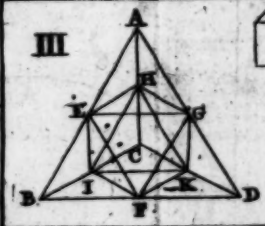

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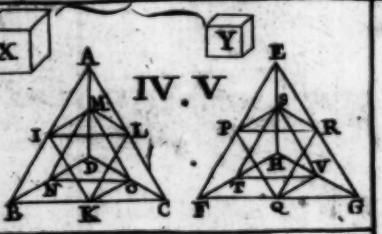

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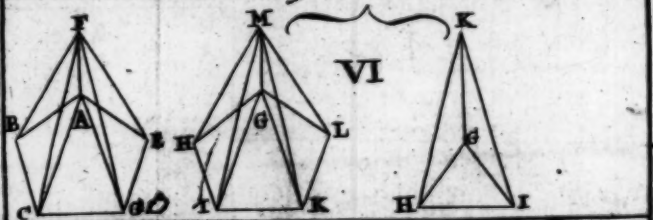
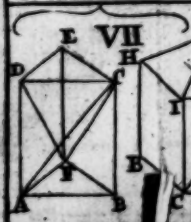
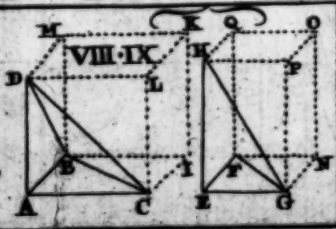


VII



VIII. IX



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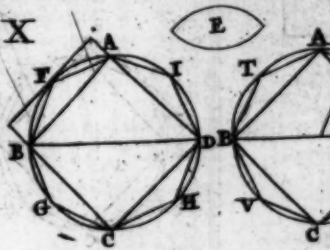

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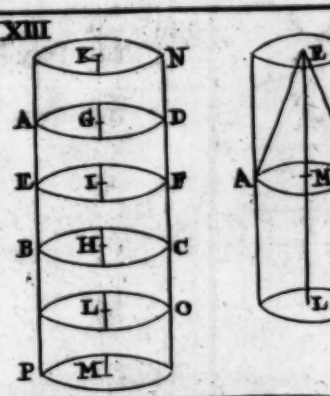
XIII



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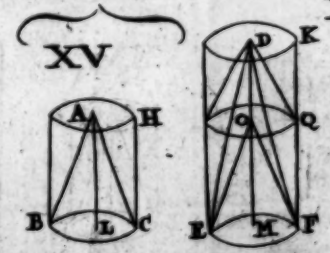

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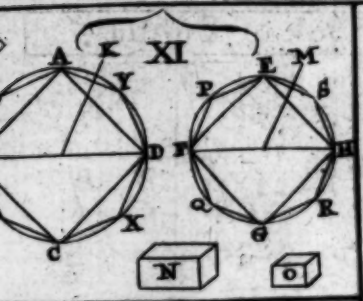

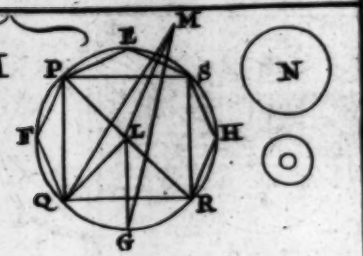

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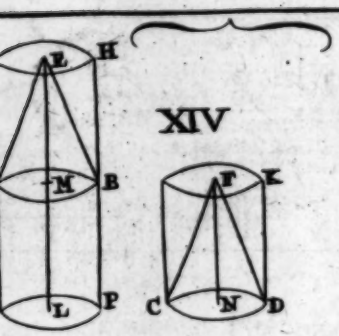

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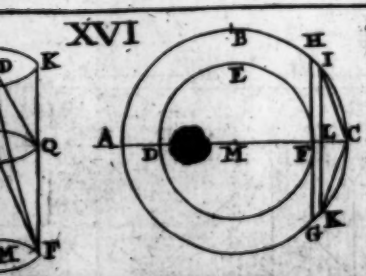

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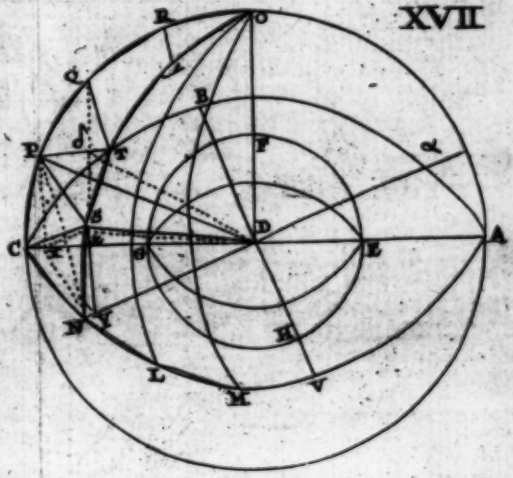

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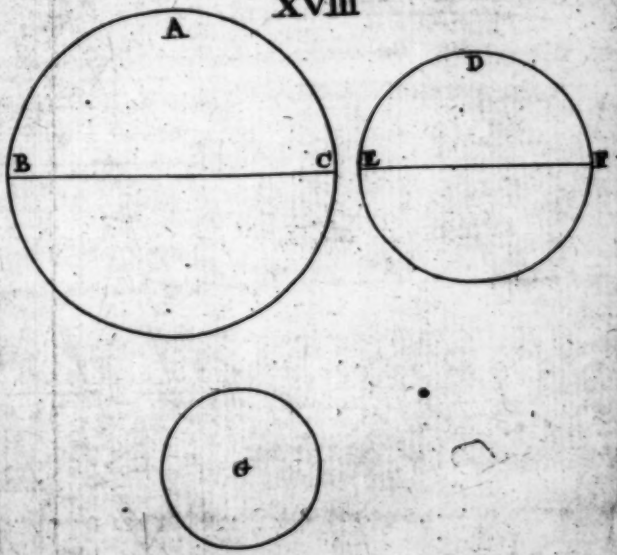
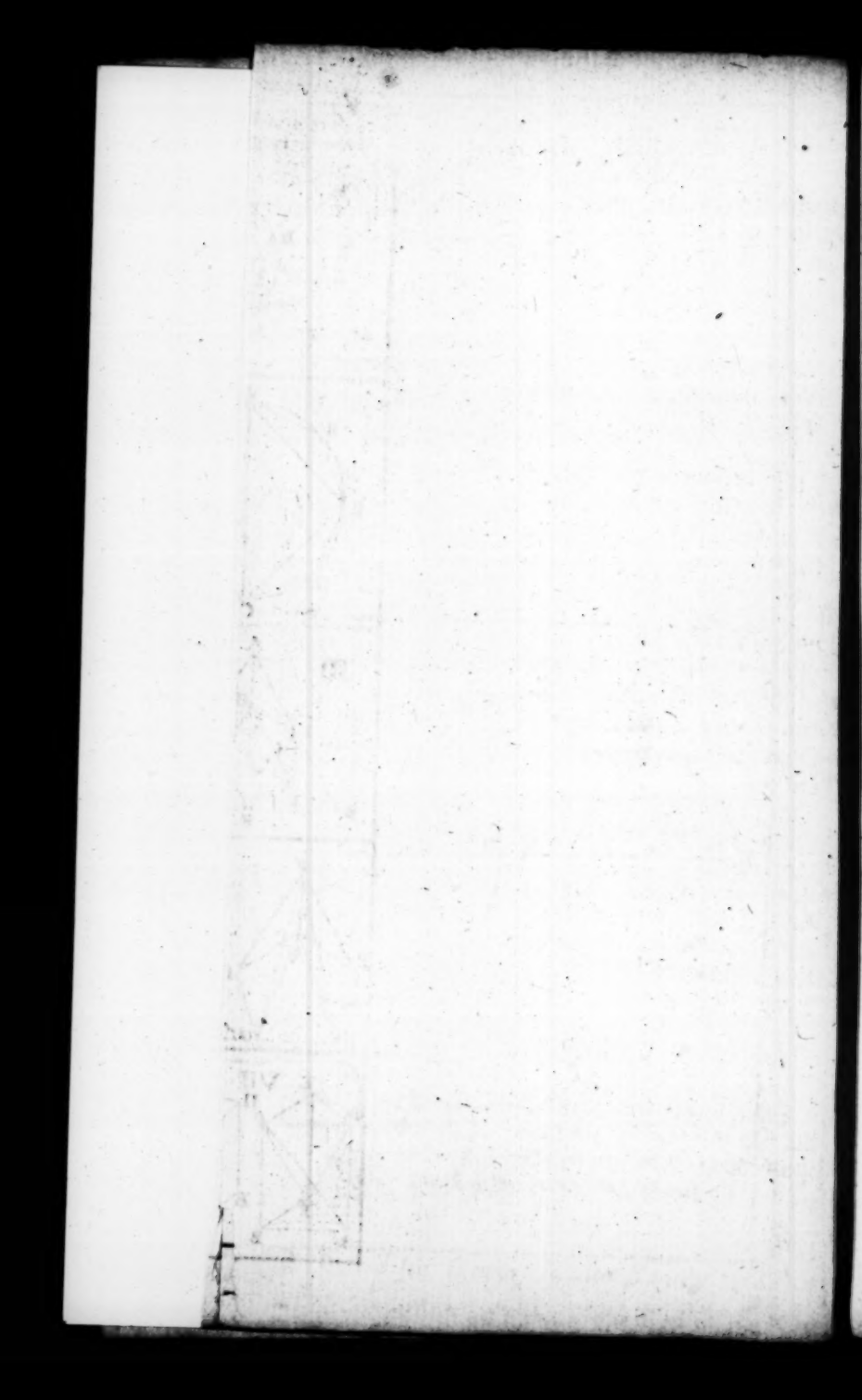


nd. of the Twelfth Book.

Tho: Cross Junior Sculp.



polyedron of the other ſphere, proportion triple of that of the ſemidiameters, [c] and ſo of the diameters of the ſpheres. [c] 15. 5.

PROP. XVIII.

Spheres BAC, EDF, are in triple proportion one to the other of that in which their diameters BC, EF, are.

Let the ſphere BAC be to the ſphere G in triple proportion of that of the diameter BC to the diameter EF, I ſay G = EDF. For if it be poſſible, let G be ⊐ EDF; and conceive the ſphere G concentrical with EDF. In the ſphere EDF [a] inſcribe a polyedron not touching the ſphere G, and a like polyedron in the ſphere BAC. Theſe polyedrons [b] are in triple proportion of the diameters BC, EF, [c] that is, of the ſphere BAC to G. [d] Conſequently the ſphere G is greater than the polyedron inſcribed in the ſphere EDF, the part than the whole.

[a] 17. 12.
[b] *cor.* 17. 12
[c] *hyp.*
[d] 14. 5.

Again, if it be poſſible, let the ſphere G be ⊏ EDF. and as the ſphere EDF is to another ſphere H, ſo let G be to BAC, [e] that is, in triple proportion of the diameter EF to BC. therefore ſince BAC [f] ⊏ H, we ſhall incurr the abſurdity of the firſt part. wherefore rather the ſphere G = EDF. Which was to be demonſtrated.

[e] *hyp. in e.*
[f] 14. 5.

Corollarium.

Hence, As one ſphere is to another ſphere, ſo is a polyedron deſcribed in that to a like polyedron deſcribed in this.

The End of the Twelfth Book.

THE THIRTEENTH BOOK OF Euclid's *Elements.*

PROP. I.

IF a right line z be divided according to extream and mean proportion (z. a :: a. e.) the ſquare of the half of the whole line z, and of the greater ſegment a, as one line, is quintuple to that which is deſcribed of half of that whole line z.

I ſay Q. $a + \frac{1}{2}z = 5$ Q. $\frac{1}{2}z$. [a] that is, $aa + \frac{1}{4}zz + za = zz + \frac{1}{4}zz$. [b] or $aa + za = zz$. For $ze + za$ [c] $= zz$. and ze [d] $= aa$. [e] therefore $aa + za = zz$. Which was to be demonſtrated.

[a] 4. 2.
[b] 3. *ax.* 1.
[c] 2. 2.
[d] *hyp. and* 16. 6.
[e] 2. *ax.* & 1. *ax.*

PROP.

PROP. II.

See the I. *Scheme.*

If a right line $\frac{1}{2}z + a$ *be in power quintuple to a ſegment of it ſelf* $\frac{1}{2}z$. *the line double of the ſaid ſegment* (z) *being divided according to extream and mean proportion, the greater ſegment is* (*a*) *the other part of the right line at firſt given* $\frac{1}{2}z + a$.

I ſay z. a :: a. e. For becauſe by the hyp. * $aa + \frac{1}{4}zz + za = zz + \frac{1}{4}zz$; or $aa + za = zz$ [a] $= ze + za$, [b] thence ſhall aa be $= ze$. [c] wherefore z. a :: a. e. Which was to be demonſtrated.

* 4. 2.
a 2. 2.
b 3. *ax.* 1.
c 17. 6.

PROP. III.

If a right line z *be divided according to extream and mean proportion* ($z.a :: a.e$.) *the line made of the leſs ſegment* e *and half of the greater ſegment* a, *is in power quintuple to the ſquare, which is deſcribed of the half line of the greater ſegment* a.

I ſay $Q:e + \frac{1}{2}a = 5Q:\frac{1}{2}a$. [a] that is $ee + \frac{1}{4}aa + ea = aa + \frac{1}{4}aa$. [b] or $ee + ea = aa$. For $ee + ea$ [c] $= 2e$ [d] $= aa$. Which was to be demonſtrated.

a 4. 2.
b 3. *ax.*
c 3. 2.
d *hyp. and* 17. 6.

PROP. IV.

If a right line z *be cut according to extream and mean proportion* ($z.a :: a.e$) *the ſquare made of the whole line* z, *and that made of the leſſer ſegment* e, *both together, are triple of the ſquare made of the greater ſegment* a.

I ſay zz -|- ee = 3 aa. [a] or aa -|- ee -|- 2 ae -|- ee = 3 aa. For ae -|- ee [b] = ze [c] = aa. [d] therefore aa -|- 2 ae -|- 2 ee = 3 aa. Which was to be demonſtrated.

[a] 4. 1.
[b] 3. 2.
[c] 17. 6.
[d] 2 *ax.*

PROP. V.

If a right line AB be cut according to extream and mean proportion in C, and a line AD, equal to the greater ſegment AC, added to it, the whole right line DB is divided according to extream and mean proportion; and the greater ſegment is the right line AB given at the beginning.

For becauſe AB. AD [a] :: AC. CB. and by inverſion AD. AB :: CB. AC. therefore by compoſition DB. AB :: AB. AC (AD.) Which was to be demonſtrated.

[a] *hyp.*

Scholium.

But if BD. BA :: BA. AD. then ſhall be BA. AD :: AD. BA—AD. For by diviſion is BD—BA (AD). BA :: BA—AD. AD. therefore inverſely BA. AD :: AD. BA—AD.

PROP. VI.

If a rational right line AB be cut according to extream and mean proportion in C, either of the ſegments (AC, CB) is an irrational line of that kind which is called apotome or reſidual.

To the greater ſegment AC [a] add AD = ½ AB. [b] therefore DCq = 5 DAq. [c] therefore DCq ⊓ DAq. conſequently [d] ſince AB, [e] and ſo the half thereof DA are ῥ, likewiſe DC is ῥ. But becauſe 5. 1 :: not Q. Q. [f] thence is DC ⊓ DA. [g] therefore DC—AD, that is, AC, is

[a] 3. 1.
[b] 1. 13.
[c] 6. 10.
[d] *hyp.*
[e] *ſcb.* 12. 10
[f] 9. 10.
[g] 74. 10.

is a reſidual line. Further, becauſe ACq [h] = AB × BC, and AB is ρ, [k] likewiſe BC is a reſidual line. Which was to be demonſtrated. [h] 17. 6. [k] 98 10.

PROP. VII.

If three angles of an equilateral Pentagone ABCDE, whether they follow in order, (EAB, ABC, BCD,) or not, (EAB, BCD, CDE) be equal, the pentagone ABCDE ſhall be equiangular.

Let the right lines BE, AC, BD, be ſubtended to the equal angles in order.

Being the ſides EA, AB, EC, CD, and the included angles [a] are equal, [b] therefore ſhall the baſes BE, AC, BD, [c] and the angles AEB, ABE, BAC, BCA, be equal. [d] wherefore BF = FA, [e] and conſequently FC = FE; therefore the triangles FCD, FED, are equilateral one to the other: [f] whence the angle FCD = FED. [g] conſequently the ang. AED = BCD. In like manner the ang. CDE is equal to the reſt; wherefore the pentagone is equiangular. Which was to be demonſtrated. [a] *hyp.* [b] 4. 1. [c] 4 & 5. 1. [d] 6. 1. [e] 3 *ax.* 1. [f] 8. 1. [g] 2 *ax.* 1.

But if the angles EAB, ECD, CDE, which are not in order, be ſuppoſed equal, [h] then ſhall the ang. AEB be = BDC, and BE = BD. [k] and thence the ang. BED = BDE. [l] conſequently the whole ang. AED = CDE. therefore becauſe the angles A, E, D, in order, are equal, as before, the pentagone ſhall be equiangular. Which was to be demonſtrated. [h] 4. 1. [k] 5. 1. [l] 2. *ax.*

PROP. VIII.

If in an equilateral and equiangular Pentagone ABCDE, two right lines BD, CE, ſubtend two angles BCD, CDE, following in order, thoſe lines do cut one another according to extream and mean proportion, and their greater ſegments BF or EF are equal to the ſide of the Pentagone BC.

a 14. 4. b 28. 3. c 27. 3. d 32. 1. e 33. 6. f 6. 1. g 27. 3. h 4. 6.

a Deſcribe about the pentagone the circle ABD. b The arch ED is = BC, c therefore the angle FCD = FDC. d therefore the ang. BFC = 2 FCD (FCD + FDC.) But the arch BAE b is = 2 ED, and conſequently the angle BCF e = 2 FCD = BFC. f wherefore BF = BC. Which was to be demonſtrated. Moreover becauſe the triangles BCD, FCD, are g equiangular, h therefore BD. DC. (BF.) :: CD. (BF.) FD. and likewiſe EC. EF :: EF. FC. Which was to be demonſtrated.

PROP. IX.

If the ſide of an Hexagone BE, and the ſide of a Decagone AE, both deſcribed in the ſame circle ABC, be added together, the whole right line AE is cut according to extream and mean proportion (AE. BE :: BE. AB,) and the greater ſegment thereof is the ſide of the Hexagone BE.

Draw the diameter ADC, and joyn the right lines DB. DE. Becauſe the ang. BDC a = 4 EDA, and the ang. BDC b = 2 DBA (DAB + DBA) thence ſhall DBA (b BDE + EED) c be = 2 EDA d = 2 EDE. whence the ang. DBA or DAB e = ADE. Therefore the Triangles ADE, ADB, are equiangular:

a hyp. and 17. 3. b 32. 1. c 7. ax. 1. d 5. 1. e 1 ax. 1.

f where-

f wherefore AB. AD (=BE) :: AD (BE). AB. f 4. 6.
Which was to be demonſtrated. g cor. 15. 4.

Corollarium.

Hence, If the ſide of a hexagone in a circle be cut according to extream and meaſ proportion; the greater ſegment thereof ſhall be the ſide of the Decagone in the ſame circle.

PROP. X.

If an equilateral Pentagone ABCDE be deſcribed in a circle ABCE, the ſide of the pentagone AB containeth in power both the ſide of a hexagone FB, and the ſide of a decagone AH deſcribed in the ſame circle.

Draw the diameter AG, and biſect equally the arch AH in K. and draw FK, FH, FB, BH, HM.

a 28. 3. and 3 ax.

The ſemicircle AG — the arch AC a = AG — AD. that is, the arch CG = GD b = AH = HB. therefore the arch BCG = 2 BHK; c and ſo the ang. BFG = 2 BFK. d but the ang. BFG = 2 BAG. e therefore the ang. BFK = BAG. Wherefore the triangles BFM, FAB, f are equiangular. g whence AB. BF :: BF. BM. h therefore AB × BM = BFq. Moreover the ang. AFK k = HFK, and FA = FH. m wherefore AL = LH, m and the angles FLA, FLH are equal, and ſo right angles, therefore the ang. LHM m = LAM n = HBA: therefore the triangles AHB, AMH, o are equiangular. wherefore AB. AH :: AH. AM. q therefore AB × AM = AHq. So that ſeeing ABq r = AB × BM + AB × AM. f thence

b hyp. & 7. ax.
c 33. 6.
d 20. 3.
e 1 ax. 1.
f 32. 1.
g 4. 6.
h 17. 6,
k 27. 3.
m 4. 1.
n 27. 3.
o 32. 1.
p 4. 6.
q 17. 6
r 2. 2.

A q f 2. ax

ABq = BFq + AHq. Which was to be demonſtrated.

Corollaria.

1. Hence, a right line (FK) which being drawn from the center (F) divides an Arch (HA) into two equal ſegments, does alſo divide the right line (HA) ſubtending that arch, perpendicularly into two equal ſegments.

2. The diameter of a circle (AG) drawn from any angle (A) of a pentagone, does divide equally in two both the arch (CD) which the ſide of the pentagone oppoſite to that angle ſubtends, and alſo the oppoſite ſide it ſelf (CD) and that perpendicularly.

Scholium.

Here, according to our promiſe, we ſhall lay down a ready praxis of the 11. Prop. *of the* 4. *Book.*

Probl.

To find out the ſide of a pentagone to be inſcribed in a circle ADB.

Draw the diameter AB, to which erect a perpendicular CD at the center C, divide CB equally in E. and make EF = ED. then DF ſhall be the ſide of the pentagone.

For BF × FC + ECq [a] = EFq [b] = EDq [c] = DCq + ECq. [d] therefore BF × FC = DCq or ECq. [e] wherefore BF. EC :: BC. FC. therefore ſince BC is the ſide of a hexagone, [f] FC ſhall be the ſide of a decagone. Conſequently DF [h] = √ DCq + FCq [g] is the ſide of a pentagone. Which was to be done.

a 6. 2.
b *conſtr.*
c 47. 1.
d 3. *ax.*
e 17. 6.
f 9. 13.
g 10. 13.
h 47. 1.

PROP.

PROP. XI.

If in a circle ABCD, whoſe diameter is rational AG, an equilateral pentagone be inſcribed ABCDE; the ſide of the pentagone AB is an irrational line, of that kind which is called a minor line.

Draw the diameter BFH, and the right lines AC, AH; and * make FL = ¼ of the ray FH; and CM = ¼ CA. * 10. 6.

Becauſe the angles AKF, AIC, are [a] right angles, and CAI commune, the triangles AKF, AIC, are [b] equiangular: [c] therefore CI. FK [c] :: CA. FA (FB) [d] :: CM. FL. therefore by permutation FK. FL :: CI. CM [d] :: CD. CK (2 CM) and ſo by [e] compoſition CD + CK. CK :: KL. FL. [f] conſequently Q: CD + CK (5 CKq). CKq :: KLq. FLq. [g] therefore KLq = 5 FLq. wherefore if BH (ρ) be taken 8, FH ſhall be 4, FL 1, and FLq 1, BL 5, and BLq 25, KLq 5. by which it appears, that BL and KL are ρ [h] ☐, [k] and ſo BK is a reſidual, and KL its congruent or adjoyning line. But being BLq — KLq = 20, [l] thence BL ☐ √ BLq — KLq. [m] whence BK ſhall be a fourth reſidual line. Therefore becauſe ABq [m] is = HB × BK, [n] ſhall AB be a minor line. Which was to demonſtrated.

[a] *cor.* 10. 13
[b] 32. 1.
[c] 4. 6.
[d] 15. 5.
[e] 18. 5.
[f] 22. 6.
[g] 1. 13.
[h] 9. 10.
[k] 74. 10.
[l] 9. 10.
[m] *cor.* 8. 6. *and* 17. 6.
[n] 95. 10.

PROP. XII.

If in a circle ABEC an equilateral triangle ABC be inſcribed, the ſide of that triangle AB is in power triple to the line AD drawn from D the center of the circle to the circumference.

The Diameter being extended to E, draw BE. Becauſe the arch BE [a] = EC, the arch [a] *cor.* 10. 13

BE

b *cor.* 15. 4. BE is the ſixth part of the circumference. b therefore BE = DE. hence ABq
c 4. 2. = 4 DEq (4 BEq)
d 47. 1. d = ABq + BEq (+ ADq.) e conſequently
e 3. *ax.* 1. ABq = 3 ADq. Which was to be demonſtrated.

Corollaria.

f *cor.* 8. 6. 1. AEq. ABq :: 4. 3.
and 22. 6. 2. ABq. AFq :: 4. 3. f For ABq. AFq :: AEq. ABq.
g *cor.* 15. 4. 3. DF = FE. For the triangle EBD g is
h *cor.* 3. 3. equilateral, h and BF perpendicular to ED. h therefore EF = FD.
4. Hence, AF = DE + DF = 3 DF.

PROP. XIII.

To deſcribe a pyramid EGFI, and comprehend it in a ſphere given: and to demonſtrate that the diameter of the ſphere AB is in power ſeſquialtera of the ſide EF of the pyramid EGFI.

About AB deſcribe the ſemicircle ADB;
a 10. 6. a and let AC be = 2 CB. from the point C
erect the perpendicular line CD; and joyn
AD, DB. then at the interval of the ray HE
b *cor.* 15. 4. = CD deſcribe the circle HEFG. b wherein
inſcribe the equilateral triangle EFG. from
c 12. 11. H c erect IH = CA perpendicular to the
d 3. 1. plane EFG. produce IH to K, d ſo that IK
= AB; and joyn the right lines IE, IF, IG.
Then EFGI ſhall be the pyramid required.

For becauſe the angles ACD, IHE, IHF, IHG, e are right angles; and CD, HE, HF, HG e equal, e and IH = AC; f therefore AD,
e *conſtr.* IE, IF, IG, ſhall be equal among themſelves.
f 41. 1. But being AC (2 CB). CB :: ACq. CDq.
g 20. 6. thence

thence ſhall ACq be = 2 CDq. therefore ADq [c] = ACq + CDq [d] = 3 CDq = 3 HEq [k] = EFq. [l] therefore AD, FF, LE, IF, EG are equal, and ſo the pyramid EFGI is equilateral. But if the point C be placed upon H, and AC upon HI, the right lines AB, HI, [m] ſhall agree, as being equal. Wherefore the ſemicircle ADB being drawn about the axis AB or IK, [n] ſhall paſs by the points E, F, G, * and ſo the pyramid EFGI ſhall be inſcribed in a ſphere. Which was to be done.

Alſo it is manifeſt that BAq. ADq [o] :: BA. AC [p] :: 3. 2. Which was to be demonſtrated.

[c] 2. *ax.*
[d] 12. 13.
[k] 1. *ax.* 1.
[l] 8. *ax.*
[m] 15. *def.* 1.
[n] 31. *def.* 11
[o] *cor.* 8. 6.
[p] *conſtr.*

Corollaria.

1. ABq. HEq :: 9. 2. For if ABq be put 9, then ACq (EFq) ſhall be 9. [q] conſequently HEq ſhall be 2.

2. If L be the center, then ſhall AB. LC :: 6. 1. For if AB be put 6, then AL ſhall be 3, [r] and thence AC 4. wherefore LC ſhall be 1. Hence

3. AB. HI :: 6. 4 :: 3. 2. whence

4. ABq. HIq :: 9. 4.

[q] 12. 13.
[r] *conſtr.*

PROP. XIV.

To deſcribe an Octaedron KEFGDL, and comprehend it in the given ſphere, wherein a pyramid is: and to demonſtrate that AH the diameter of the ſphere is in power double of AC the ſide of the Octaedron.

About AH deſcribe the ſemicircle ACH, and from the center B erect the perpendicular BC. draw AC, HC. then upon ED = AC [a] make the ſquare EFGD, whoſe diameters DF, EG, cut

[a] 46. 1.

cut in the center I. from I draw IL = AB
[b] 12. 11. [b] perpendicular to the plane EFGD, produce
[c] 3. 1. IL, [c] till IK = IL. and joyn KE, KF, KG, KD, LE, LF, LG, LD; then ſhall KEFGDL be the Octaedron required.

For AB, BH, FI, IE, &c. being ſemidiameters of equal ſquares are equal one to the
[d] 4. 1. other. [d] whence the baſes LF, LE, FE, &c. of the right angled triangles LIE, LIF, FIE, &c. are equal, and conſequently the eight triangles LFE, LFG, LGD, LDE, KEF, KFG, KGD,
[e] 27. *def.* 11 KDE, are equilateral, [e] and make an Octaedron, which may be inſcribed in a ſphere, whoſe center is I, and IL or AB the radius.
[f] *conſtr.* (becauſe AB, IL, IF, IK, &c. [f] are equal.) Which was to be done. Moreover, it is evi-
[g] 47. 1. dent, that AHq (LKq) [g] = 2 ACq (2 LDq.) Which was to be demonſtrated.

Corollaria.

1. Hence it is manifeſt, that in the octaedron the three diameters EG, FD, LK do cut one the other perpendicularly in the center of the ſphere.

2. Alſo that the three planes EFGD, LEKG, LFKD, are ſquares, cutting one another perpendicularly.

3. The Octaedron is divided into two like and equal pyramids EFGDL, and EFGDK, whoſe common baſe is the ſquare EFGD.

15. 11. 4. Laſtly, it follows that the oppoſite baſes of the octaedron are parallel one to the other.

PROP.

PROP. XV.

To describe a cube EFGHIKLM, and comprehend it in the same sphere wherein the former figures were; and to demonstrate that AB the diameter of the sphere is in power triple to EF the side of that cube.

Upon AB describe a semicircle ACB; [a] and make AB = 3 DA; from D raise the perpendicular DC, and joyn BC and AC. Then upon EF = AC [b] make the square EFGH, upon whose plane let the right lines EI, FK, HM, GL, stand perpendicular, being equal to EF, and connect them with the right lines IK, KL, LM, IM. The solid EFGHIKLM is a cube; as is sufficiently apparent from the construction.

In the opposite squares EFKI, HGLM, draw the diameters EK, FI, HL, MG. by which let the planes EKLH, FIMG be drawn, cutting one another in the line NO. which [c] shall divide equally in two parts the diameters of the cube EL, FM, GI, HK, in P the center of the cube. [d] therefore P shall be the center of a sphere passing by the angular points of the cube. Moreover, ELq [e] = EKq + KLq [e] = 3 KLq, [f] or 3 ACq. but ABq. ACq [g] :: BA. DA [f] :: 3.1. [h] therefore AB = EL; wherefore we have made a cube, *&c.* Which was to be done.

a 10. 6.
b 46. 11.
c *cor.* 39. 11
d 15 *def.* 11 *and* 14 *def.* 11.
e 47. 1.
f *constr.*
g *cor.* 8. 6.
h 14. 5.

Corollaria.

1. Hence it is manifest, that all the diameters of the cube are equal one to another, and do equally bisect one another in the center of the sphere. And by the same means the right lines which conjoyn the centers of the oppo-

site

site squares are equally biſected in the ſame center.

[k] 47. 1. [l] 13. 15. [m] 15. 13. 2. The diameter of a ſphere containeth in power the ſide of a tetraedron and of a cube, *viz.* ABq [k] = [l] BCq + [m] ACq.

PROP. XVI.

To deſcribe an Icoſaedron ZGHIKFIVXRST, and encompaſs it in the ſphere, wherein were contained the foreſaid ſolids; and to demonſtrate that FG the ſide of the Icoſaedron is that irrational line, which is called a minor line.

Upon AB the diameter of a ſphere deſcribe [a] 10. 6. the ſemicircle ADB; and [a] make AB = 5 BC. then from C erect CD perpendicular, and draw AD and BD. At the diſtance EF = BD [b] 11. 4. deſcribe the circle EFKNG; [b] wherein in- [c] 12. 11. ſcribe the equilateral pentagone FKNG. Divide equally in two parts the arches FG, GH, &c. and joyn the right lines FL, LG, &c. being the ſides of a decagone. Then [c] erect EQ, LR, MS, NT, OV, PX equal to EF, and perpendicular to the plane FKNG; and connect RS, ST, TV, VX, XR; as alſo FX, FR, GR, GS, HS, ST, HT, IT, IV, KV, KX. Laſtly, produce EQ, and take QY = FL, and EZ = FL. and conceive the right lines ZG, ZH, ZI, ZK, ZF to be drawn; as alſo YV, YX, YR, YS, YT. Then I ſay the Icoſaedron required is made.

For becauſe EQ, LR, MS, NT, OV, PX, [d] *conſtr.* are [d] equal and [e] parallel, alſo thoſe lines that [e] 6. 11. joyn them EL, QR, LM, QS, LN, QT, EO, [f] 33. 1. QV, EP, QX, [f] are equal and parallel. And thence likewiſe LM (or FG) RS, MN, ST, &c. [g] 15. 11. are equal one to the other. [g] therefore the plane drawn by LR, LM, &c. is diſtant equally from

from the plane paſſing by QR, QS, &c. [h] and the circle QXRSTV drawn from the center Q is equal to the circle EPLMNO; and RSTVX is an equilateral pentagone. But EF, EG, EH, &c. and QX, QR, QS, &c. being conceived to be drawn; then becauſe FRQ [k] = FLq + LRq, [l] or EFq [m] = FGq, [n] therefore FR, FG, and ſo all RS, EG, FR, RG, GS, GH, &c. ſhall be equal one to the other. and conſequently the ten triangles RFX, RFG, RGS, &c. are equilateral and equal. Moreover, becauſe XQY is a [o] right angle, therefore XYq [p] = QXq + QYq [q] = VXq or FGq. wherefore XY, VX, and likewiſe YV, YT, YS, YR, ZG, ZH, &c. are equal. Therefore other ten triangles are made, equilateral and equal both to one another, and to the ten former; and ſo an Icoſaedron is made.

Moreover, divide equally EQ in α, draw the right lines αF, αX, αV; and becauſe QX [r] = QV, and αQ the common ſide, and FQX, EQV are right angles, [s] therefore ſhall αX be = αV; and by the ſame reaſon all the lines αX, αR, αS, αT, αV, αF, αG, αH, αI, αK, are equal. But becauſe ZQ.QE [t] :: QE.ZE. therefore Zαq [u] = 5 Eαq [x] = EQq (EFq) + Eαq [y] = αFq. therefore Zα = αF. [z] in like manner αF = Yα. therefore the ſphere, whoſe center is α, and αF the ray, ſhall paſs by the 12 angular points of the Icoſaedron.

Laſtly, becauſe Zα.αE :: ZY.QE; [a] and ſo Zαq.αEq :: ZYq.QEq. [b] therefore ZYq = 5 QEq, or 5 BDq = : but ABq.BDq [c] :: AB. BC :: 5.1. [d] therefore ZY = AB. Which was to be done.

Therefore if AB be put ρ, [e] then EF = √ ABq ſhall be alſo ρ. and conſequently FG the ſide

[h] 1. *def.* 3.
[k] 47. 1.
[l] *conſtr.*
[m] 10. 13.
[n] *ſch.* 48. 1. *and* 1. *ax.*
[o] *cor.* 14. 11
[p] 47. 1.
[q] 10. 13.
[r] 15. *def.* 11.
[s] 4. 1.
[t] 9. 13.
[u] 3. 13.
[x] 4. 2.
[y] 47. 1.
[z] 15. 5.
[a] 22. 6.
[b] 14. 5.
[c] *cor.* 8. 6.
[d] 1 *ax.* 1.
[e] *ſch.* 12. 10

ſide of the pentagone, and likewiſe of the Icoſaedron, [f] is a minor line. Which was to be demonſtrated.

[f] 11. 13.

Corollaria.

1. From hence is inferred, that the diameter of the ſphere is in power quintuple of the ſemidiameter of the circle encompaſſing the five ſides of the Icoſaedron.

2. Alſo it is manifeſt, that the diameter of the ſphere is compoſed of the ſide of a hexagone, that is, of the ſemidiameter, and two ſides of the decagone of a circle encompaſſing ths five ſides of the Icoſaedron.

3. It appears likewiſe, that the oppoſite ſides of an Icoſaedron, ſuch as RX, HI, are parallels. For RX [a] is parall. to LP. [b] parall. to HI.

[a] 33. 1.
[b] *ſch.* 26. 3.

PROP. XVII.

To deſcribe a Dodecaedron, and comprehend it in the ſphere wherein the former figures were comprehended: and to demonſtrate that the ſide RS of the Dodecaedron is an irrational line of that ſort which is called an apotome or reſidual line.

Let AB be a cube inſcribed in the given ſphere, and let all the ſides thereof be divided equally in the points E, H, F, G, K, L, &c. and joyn the right lines KL, MH, HG, EF. [a] make HI. IQ :: IQ. QH; and take NO, NP, = IQ. then erect OR, PS, perpendicular to the plane DB, and QT to the plane AC; and let OR, PS, QT, be equal to IQ, NO, NP. whence DR, RS, SC, CT, DT, being connected, DRSCT ſhall be a pentagone of the dode-

[a] 30. 6.

caedron

caedron required. For draw NV parallel to OR, and having drawn NV out as far as the center of the cube X, joyn the right lines DS, DO, DP, CR, CP, HV, HT, RX. Because DOq[a] = DKq ([b] KNq) + KOq[c] = 3 ONq (3 ORq) [d] thence DRq = 4 ORq[e] = OPq, or RSq. therefore DR = RS. By the same reason DR, RS, SC, CT, TP, are equal. But because OR [f] is = and [g] parallel to PS, therefore RS, OP, and [h] consequently RS, DC shall be also parallels. [h] therefore these with them that conjoyn them DK, CS, VH, are in one and the same plane. Moreover, because HI. IQ [k] :: IQ. (TQ.) QH [k] :: HN. NV. and both TQ, HN, and QH, NV [k] are perpendicular to the same plane, [l] and so likewise parallels, [m] THV shall be a right line. [n] therefore the Trapezium DRSC, and the triangle DTS are in one plane extended by the right lines DC, TV. [o] therefore DCTSR is a pentagone, and that also equilateral, by what is shewn already. Furthermore, because PK. KN :: KN. NP; and DSq [p] = DPq + PSq (PNq) = [p] DKq + PKq + NPq, [q] thence DSq = DKq + 3 KNq = 4 DKq (4 DHq) [r] = DCq. therefore DS = DC. whence the triangles DRS, DCT, are equilateral one to another. [s] therefore the angle DRS = DTC, and likewise the ang. CSR = DCT. therefore the pentagone DTCSR is also equiangular. Moreover, because AX, DX, CX, &c. are semidiameters of the cube, [t] thence is XN = IH or KN, [u] and so XV = KP; wherefore because RVX, is a [x] right angle, [z] thence RXq = XVq + RVq (NPq) = KPq + NPq [a] = 3 KNq [b] = AXq or DXq, &c. therefore RX. AX, DX, and by the same reason XS, XT, AX, are equal

[a] 47. 1.
[b] 7. *ax.* 1.
[c] 4. 13.
[d] 47. 1.
[e] 4. 2.
[f] *constr.*
[g] 33. 1.
[h] 9. 11.
[k] 7. 11.
[l] 6. 11.
[m] 32. 6.
[n] 1 & 2. 11
[o] 5. 13.
[p] 47. 1.
[q] 1. *ax.* 2. & 4. 13.
[r] 4. 2.
[s] 8. 1.
[t] 15. 13.
[u] 1. *ax.* 1.
[x] 29. 1.
[z] 47. 1.
[a] 4. 13.
[b] 15. 13.

equal one to another. And if by the ſame method, whereby the pentagone DTCSR was made, twelve like pentagones, touching the twelve ſides of the cube, be made, they ſhall compoſe a Dodecaedron; and a ſphere paſſing by their angular points, whoſe ray is AX or RX, ſhall comprehend that Dodecaedron. Which was to be done.

c *conſtr.* d 15. 5. Laſtly, becauſe KN . NO c :: NO . OK. d thence KL . OP :: OP . OK + PL. Therefore if AB the diameter of the ſphere be ſuppoſed $\dot{p}$, then ſhall KL e $= \sqrt{\frac{AB}{3}}$ f be alſo $\dot{p}$. g whence OP or RS the ſide of the dodecaedron ſhall be a reſidual line. Which was to be demonſtrated.

e 15. 13. f *ſch.* 12. 10. g 6. 13.

Corollaria.

From this demonſtration it follows, 1. That if the ſide of a cube be cut in extream and mean proportion, the greater ſegment ſhall be the ſide of the dodecaedron deſcribed in the ſame ſphere.

2. If the leſſer ſegment of a right line, cut in extream and mean proportion, be the ſide of the dodecaedron, the greater ſegment ſhall be the ſide of the cube inſcribed in the ſame ſphere.

3. It is manifeſt alſo, that the ſide of the cube is equal to the right line which ſubtendeth the angle of a pentagone of the dodecaedron inſcribed in the ſame ſphere.

PROP.

PROP. XVIII.

To find out the sides of the precedent five figures, and compare them together.

Let AB be the diameter of the sphere given, and AEB the semicircle, and let AC be[a] $= \frac{1}{2}$ AB, and AD[b] $= \frac{1}{3}$ AB. then erect the perpendiculars CE, DE, and BG = AB. joyn AF, AE, BE, BF, CG; and let fall the perpendicular HI from H; and CK being taken equal to CI, from K erect the perpendicular KL, and joyn AL. Lastly, [c] make AF.AO :: AO.OF.

[a] 10. 1.
[b] 10. 6.
[c] 30. 6.

Therefore 3. 2[d] :: AB.BD[e] :: ABq.BFq. the side of a Tetraedron. and 2. 1 :: [a] AB.AC :: ABq.BBq [f] the side of an Octaedron.

[d] *constr.*
[e] *cor.* 8. 6.
[f] 14. 13.

Also 3. 1[d] :: AB.AD[e] :: ABq.AFq, [g] the side of an Hexaedron.

[g] 15. 13.

Moreover, because AF.AO[h] :: AO.OF. [k] thence shall AO be the side of a Dodecaedron. Lastly, BG. (2 BC.) BC[l] :: HI.IC. [m] therefore HI = 2 CI[n] = KI. therefore HIq[o] = 4 CIq. consequently CHq[p] = 5 CIq. [q] therefore ABq = 5 KIq. [r] therefore KI or HI is a ray of a circle enclosing the pentagone of an Icosaedron; and AK or IB[r] is the side of a decagone inscribed in the same circle. [s] whence AL shall be the side of a pentagone, [t] and also the side of an Icosaedron. Whereby it appears, that BF, BE, AE are ρ ⊐. and AL, AO ρ ⊐, and BF ⊏ BE, and BE ⊏ AF, and AF ⊏ AO. And because 3 AFq = ABq[u] = 5 KLq, and AF × AO ⊏ AF × OF, [x] and so AF × AO + AF × OF ⊏ 2 AF × OF, [y] that is, AFq ⊏ [z] 2 AOq. [a] thence shall 3 AFq (5 KLq) be ⊏ 6 AOq; consequently KL ⊏ AO; and much rather AL ⊏ AO.

[h] *constr.*
[k] *cor.* 17. 13
[l] 4. 6.
[m] 14. 5.
[n] *constr.*
[o] 4. 2.
[p] 47. 1.
[q] 15. 5.
[r] *cor.* 16. 13
[s] 10. 13.
[t] 16. 13.
[u] 1. 6.
[x] 4 *ax.* 1.
[y] 1. 2.
[z] 17. 6.
[a] 47. 1.

That

That we may expreſs theſe ſides in numbers; If AB be ſuppoſed $\sqrt{60}$, then, reducing what is already ſhewn to ſupputation, BF $= \sqrt{40}$, and BE $= \sqrt{30}$, and AF $= \sqrt{20}$. Alſo AL $= \sqrt{:30 - \sqrt{180}}$ (for AK $= \sqrt{15 - \sqrt{3}}$. and KL (HI) $= \sqrt{12}$.) Laſtly AO $= \sqrt{:30 - \sqrt{500}}$ ($\sqrt{25 - \sqrt{5}}$.)

Scholium.

It is very apparent, that beſides the five aforeſaid figures, there cannot be deſcribed any other regular ſolid figure (viz. *ſuch as may be contained under ordinate and equal plane figures.*)

[a] 21. 11. [b] *See ſchol.* 32. 1.

For three plane angles at leaſt are required to the conſtituting of a ſolid angle; [a] all which muſt be leſs than four right angles. [b] but 6 angles of an equilateral triangle, 4 of a ſquare, and ſix of a hexagone, do ſeverally equal 4 right angles; and 4 of a pentagone, 3 of a heptagone, 3 of an octagone, &c. do exceed 4 right angles: Therefore only of 3, 4, or 5 equilateral triangles, of 3 ſquares, or 3 pentagones, it is poſſible to make a ſolid angle. Wherefore beſides the five above mentioned, there cannot be any other regular bodies.

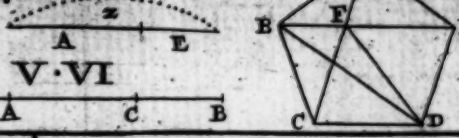
Out of P. Herigon.

The Proportions of the ſphere and the five regular figures inſcribed in the ſame.

Let the diameter of the ſphere be 2, then ſhall

The

The Thirteenth Book of Euclid. I. II. III. IV



V. VI

VIII

IX

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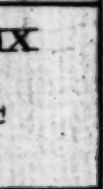
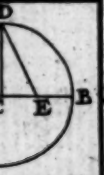
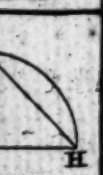
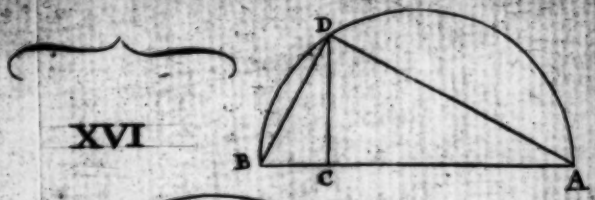
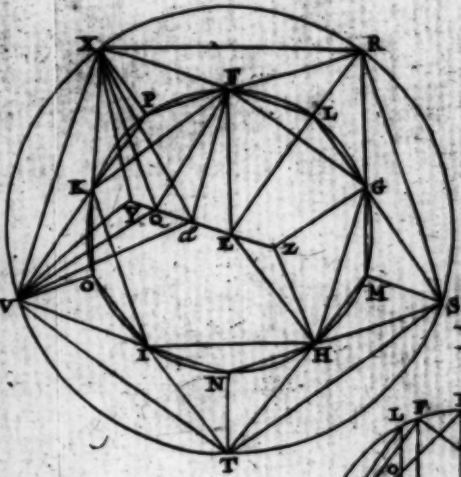
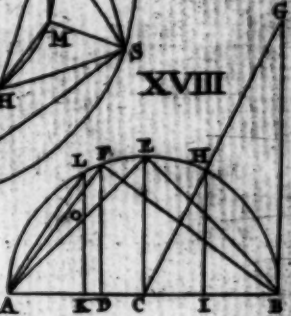


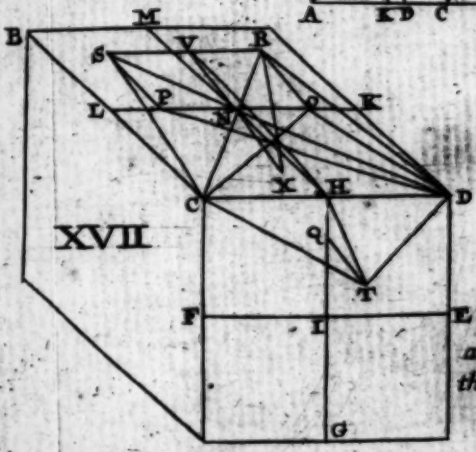






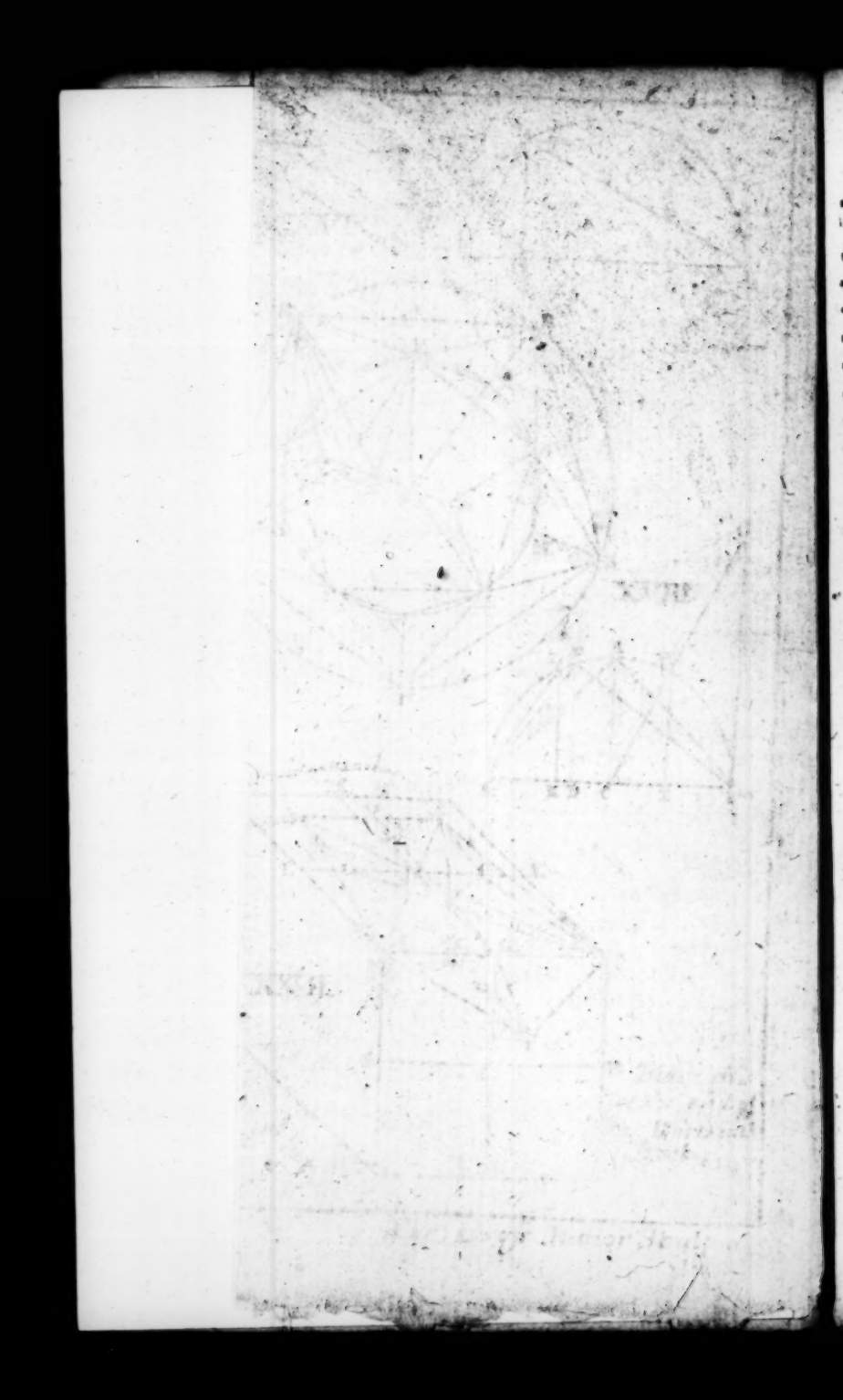






Place this at the end of the Thirteenth Book.

Tho: Cross Junior Sculp.



The periphery or circumference of the greater circle, be 6.28318.
The ſuperficies of the greater circle, 3.14159.
The ſuperficies of the ſphere, 12.56637.
The ſolidity of the ſphere, 4.1879.
The ſide of the tetraedron, 4.62299.
The ſuperficies of the tetraedron, 4.6188.
The ſolidity of the tetraedron, 0.15132.
The ſide of the hexaedron, 1.1547.
The ſuperficies of the hexaedron, 8.
The ſolidity of the hexaedron, 1.5396.
The ſide of the octaedron, 1.41421.
The ſuperficies of the octaedron, 6.9282.
The ſolidity of the octaedron, 1.33333.
The ſide of the dodecaedron, 0.71364.
The ſuperficies of the dodecaedron, 10.51462.
The ſolidity of the dodecaedron, 2.78516.
The ſide of the Icoſaedron, 1.05146.
The ſuperficies of the Icoſaedron, 9.57454.
The ſolidity of the Icoſaedron, 2.53615.

If five equilateral and equiangular figures, like theſe in the ſchemes beneath, be made of Paper, and rightly folded, they will repreſent the five regular bodies:

The End of the Thirteenth Book.

THE FOURTEENTH BOOK OF Euclid's *Elements*.

PROP. I.

A Perpendicular line DF drawn from D the center of a circle ABC to BC the side of a pentagone inscribed in the said circle, is the half of these two lines taken together, viz. *of the side of the hexagone DE, and the side of the decagone EC inscribed in the same circle ABC.*

Take FG = FE, and draw CG: [a] Then CE is = CG. therefore the ang. CGE [b] = CEG [b] = ECD. therefore the ang. ECG [c] = EDC [d] = ½ ADC [e] = ½ CED (½ ECD.) [f] consequently the angle GCD = ECG = EDC. [g] wherefore DG = GC (CE.) therefore DF = CE (DG) + EF = $\frac{DE + CE.}{2}$

Which was to be demonstrated.

a 4. 1.
b 5. 1.
c 32. 1.
d *by p. and* 33. 6.
e 20. 3.
f 7. *ax.*
g 6. 1.

PROP.

PROP. II.

If two right lines AB, DE, be cut according to extream and mean proportion (AB. AG :: AG. GB. and DE. DH :: DH. HE.) they ſhall be cut after the ſame manner, viz. *into the ſame proportions (AG. GB :: DH. HE.)*

Take BC = BG; and EF = EH. Then AB × BG is [a] = 5 AGq. wherefore ACq [b] = 4 ABG + AGq [c] = 5 AGq. In like manner ſhall DFq be = 5 DHq. [d] therefore AC. AG :: DF. DH. whence by addition AC + AG. AG :: DF + DH. DH. that is, 2 AB. AG :: 2 DE. DH. [e] conſequently AB. AG :: DE. DH; [f] whence by diviſion AG. GB :: DH. HE. Which was to be demonſtrated.

[a] 17. 6. [b] 8. 2. [c] 2. *ax.* 1. [d] 22. 5. & 22. 6. [e] 22. 5. [f] 17. 5.

PROP. III.

The ſame circle ABD comprehends both ABCDE the pentagone of a Dodecaedron, and LMN the triangle of an Icoſaedron inſcribed in the ſame ſphere.

Draw the diameter AG, and the right lines AC, CG. and let IK be the diameter of the ſphere, [a] and IKq = 5 OPq. [b] and make OP. OQ :: OQ. QP. Becauſe ACq + CGq [c] = AGq [d] = 4 FGq; and ABq [e] = FGq = CGq. [f] thence ACq + ABq = 5 FGq. moreover, becauſe CA. AB [g] :: AB. CA — AB; and OP. OQ :: OQ. QP. [h] and ſo CA. OP :: AB. OQ. [k] therefore 3 ACq ([l] IKq.) 5 OPq ([m] IKq) :: 3 ABq. 5 OQq. therefore 3 ABq = 5 OQq. But becauſe ML [n] is the ſide of a pentagone inſcribed in the circle, whoſe ray is OP, thence 15 RMq [o] = 5 MLq [p] = 5 OPq + 5

[a] *ſch.* 47. 1. [b] 30. 6. [c] 47. 1. [d] 4. 2. [e] 10. 13. [f] 1, & 3. *ax.* [g] 8. 13. [h] 2 13. & 16. 5. [k] 22. 6 & 4. 5. [l] 15. 13. [m] *conſtr.* [n] *cor.* 16 13 [o] 12. 13. [p] 10. 13.

q 15. 5. + 5 OQq = * 3 ACq + 3 AEq q = 15 EGq,
* *before.* r therefore RM = FG. f and consequently the
r 1 *ax.* circle ABD is = to the circle LMN. Which
Jch. 48. 1. was to be demonstrated.
f 1 *def.* 3.

PROP. IV.

If from F the center of a circle encompassing the pentagone of a dodecaedron ABCDE, a perpendicular line FG be drawn to one side of the pentagone CD; the rectangle contained under the said side CD and the perpendicular FG, being thirty times taken, is equal to the superficies of the Dodecaedron. Also,

If from the center L of a circle incloſing the triangle of an Icoſaedron HIK, a perpendicular line LM be drawn to one side of the triangle HK, the rectangle contained under the ſaid ſide HK, and the perpendicular LM, being thirty times taken, ſhall be equal to the ſuperficies of the Icoſaedron.

a 8. 1. Draw FA, FB, FC, FD, FE. a then ſhall
the triangles CFD, DFE, EFA, AFB, BFC, be
b 4. 1. equal. but CD × FG b = 2 triangles CFD.
c 15. 5. therefore 30 CD × GF c = 60 CFD d = 12
d 6 *ax.* pentagones ABCDE e = to the ſuperficies of
e 17. 3. the dodecaedron. Which was to be demon-
ſtrated.

f 41. 1. Draw LI, LH, LK; then HK × LM f is =
g 15. 5. 2 triang. LHK. therefore 30 HK × LM g =
h 16. 13. 60 HLK = 20 HIK h = to the ſuperficies of
the Icoſaedron. Which was to be demonſtra-
ted.

Corollarium.

k 15. 5. CD × FG. HK × LM k :: the ſuperficies of
the dodecaedron to the ſuperf. of the Icoſae-
dron.

PROP.

PROP. V.

The ſuperficies of a Dodecaedron hath to the ſuperficies of an Icoſaedron inſcribed in the ſame ſphere, the ſame proportion that H the ſide of a cube hath to AD the ſide of an Icoſaedron.

Let the circle ABCD [a] encloſe both the pentagone of a dodecaedron, and the triangle of an Icoſaedron; whoſe ſides are BD, AD. upon which from the center E let fall the perpendiculars EF, EGC; and draw CD.

Becauſe EC + CD. EC [b] :: EC. CD. thence EG. ([c] ½ EC + CD.) EF ([d] ½ EC) [e] :: EF. EG — EF (½ CD.) but H. BD [f] :: BD. H — BD. [g] therefore H . BD :: EG. EF. conſequently H × EF = BD × EG. wherefore ſince H. AD [h] :: H × EF. AD × EF. [k] thence ſhall be H. AD :: BD × EG. AD × EF [l] :: the ſuperficies of a dodecaedron to the ſuperficies of an Icoſaedron. Which was to be demonſtrated.

[a] 3. 14.
[b] 9. 13.
[c] 1. 14.
[d] *cor.* 12. 13
[e] 15. 5.
[f] *cor.* 17. 13
[g] 2. 14.
[h] 1. 6.
[k] 7. 5.
[l] *cor.* 4. 14.

PROP. VI.

If a right line AB be cut in extream and mean proportion, then as the right line BF, containing in power that which is made of the whole line AB, and that which is made of the greater ſegment AC, is to the right line E containing in power that which is made of the whole line AB, and that which is made of the leſſer ſegment BC; ſo is the ſide of the cube BG to the ſide of an Icoſaedron BK inſcribed in the ſame ſphere with the cube.

In the circle, whoſe ſemidiameter is AB, inſcribe BFGHI the pentagone of a dodecaedron, and BKL the triangle of an Icoſaedron.

[a] wherefore BG ſhall be the ſide of a cube inſcribed in the ſame ſphere. therefore BKq [b] = 3 ABq; and Eq [c] = 3 ACq. therefore BKq. Eq [d] :: ABq. ACq [e] :: BGq. BFq. wherefore by inverſion BGq. IKq :: BFq. Eq. [f] whence BG. BK :: BF. E. Which was to be demonſtrated.

[a] *cor.* 17. 13
[b] 12. 13.
[c] 4. 13.
[d] 15. 5.
[e] 2. 14.
[f] 22. 6.

PROP. VII.

A Dodecaedron is to an Icoſaedron, as the ſide of a cube is to the ſide of an Icoſaedron, inſcribed in one and the ſame ſphere.

Becauſe [a] the ſame circle comprehends both the pentagone of a dodecaedron, and the triangle of an Icoſaedron, [b] the perpendiculars drawn from the center of the ſphere to the planes of the pentagone and triangle, ſhall be equal one to another. Therefore if the Dodecaedron and Icoſaedron be conceived divided into pyramids, right lines being drawn from the center of the ſphere to all the angles, the altitudes of all the pyramids ſhall be equal one to the other. Wherefore ſince the pyramids of equal height are one to another as their baſes, and the ſuperficies of the dodecaedron is equal to twelve pentagones, and the ſuperficies of the Icoſaedron to twenty triangles, the dodecaedron ſhall be to the Icoſaedron, as the ſuperficies of the dodecaedron is to the ſuperficies of the Icoſaedron, [d] that is, as the ſide of the cube is to the ſide of the Icoſaedron.

[a] 3. 14.
[b] 47. 1.
[c] 5, *and* 6. 12.
[d] 5. 14.

PROP.

PROP. VIII.

The ſame circle BCDE comprehends both the ſquare of the cube BCDE; and the triangle of the octaedron FGH inſcribed in one and the ſame ſphere.

Let A be the diameter of the ſphere. Becauſe Aq [a] = 3 BCq [b] = 6 BIq; and alſo Aq [c] = 2 GFq [d] = 6 KFq; thence ſhall BI be = KF. [e] therefore the circle CBED = GFH. Which was to be demonſtrated.

[a] 15. 13.
[b] 47. 1.
[c] 14. 13.
[d] 22. 13.
[e] 2 *def.* 3.

The End of the Fourteenth Book.

THE

THE FIFTEENTH BOOK OF Euclid's *Elements.*

PROP. I.

IN a Cube given ABGHDCFE to deſcribe a pyramid AGEC.

From the angle C draw the diameters CA, CG, CE; and connect them with the diameters AG, GE, EA. All which are [a] equal among themſelves, as being the diameters of equal ſquares: therefore the triangles CAG, CGE, CEA, EAG are equilateral and equal; and conſequently AGEC is a pyramid, which inſiſts upon the angles of the Cube, and therefore [b] is inſcribed in the ſame. Which was to be done.

[a] 47. 1.

[b] 31. *def.* 11.

PROP.

PROP. II.

In a pyramid given ABDC to deſcribe an octaedron EGKIFH.

[a] Biſect the ſides of the pyramid in the points, E, I, F, K, G, H, which joyn with the right lines EF, FG, GE, &c. All theſe are [b] equal one to the other; conſequently the 8 triangles EHI, IHK, &c. are equilateral and equal, and ſo make [c] an octaedron deſcribed [d] in the given pyramid. Which was to be done.

[a] 10. 1.
[b] 4. 11.
[c] 27. *def.* 15.
[d] 31. *def.* 11.

PROP. III.

In a Cube given CHGBDEFA to deſcribe an octaedron NPQSOR.

Connect * the centers of the ſquares N, P, Q, S, O, R, with the twelve right lines NP, PO, QS, &c. which are [a] equal among themſelves; and ſo make 8 equilateral and equal triangles: wherefore [b] the octaedron NPQSOR [b] is inſcribed in the cube. Which was to be done.

* 8. 4.
[a] 4. 1.
[b] 31. & 27. *def.* 11.

PROP. IV.

In an Octaedron given ABCDEF, to inſcribe a Cube.

Let the ſides of the pyramid EABCD, whoſe baſe is the ſquare ABCD, be equally biſected by the right lines, LM, MN, NO, OL, which are [a] equal and [b] parallel to the ſides of the ſquare ABCD. [c] then the quadrilate-

[a] 4. 1.
[b] 2. 6.
[c] 29. *def.* 1.

lateral LMNO is a ſquare. In like manner, if the ſides of the ſquare LMNO be equally biſected in the points G, H, K, I, and GH, HK, KI, IG connected, GHKI ſhall be a ſquare. And if in the other 5 pyramids of the octaedron, the centers of the triangles be in the ſame ſort conjoyned with right lines, then other ſquares will be deſcribed like and equal to the ſquare GHKI. wherefore ſix ſuch ſquares ſhall make a cube, which ſhall be deſcribed within an octaedron, [d] being its eight angles touch the eight baſes of the octaedron in their centers. Which was to be done.

[a] 31. *def.* 11.

[d] 11.

PROP. V.

In an Icoſaedron given to inſcribe a Dodecaedron.

Let ABCDEF be a pyramid of the Icoſaedron, whoſe baſe is the pentagone ABCDE; and the centers of the triangles G, H, I, K, L; which connect with the right lines GH, HI, IK, KL, LG. Then GHIKL ſhall be a pentagone of the dodecaedron to be inſcribed.

[e] 5. 4.

For the right lines, FM, FN, FO, FP, FQ, paſſing by the centers of the triangles, [a] do equally divide their baſes into two parts: [b] therefore the right lines MN, NO, OP, PQ, QM, [c] are equal one to the other; [d] whence alſo the angles MFN, NFO, OFP, PFQ, QFM, are equal; therefore the pentagone GHIKL is equiangular, [e] and conſequently equilateral, being FG, FH, FI, FK, FL [f] are equal. And if in the other eleven pyramids of the Icoſaedron, the centers of the triangles be in like ſort

[a] *cor.* 3. 3.

[b] 4. 1.

[c] 4. 1.

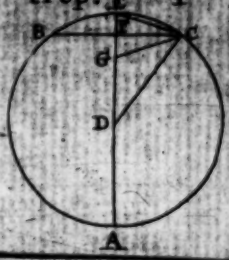
[d] 8. 1.

[e] 4. 1.

[f] 12. 13.

The Fourteenth Book of Euclid.

Prop. I



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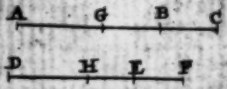

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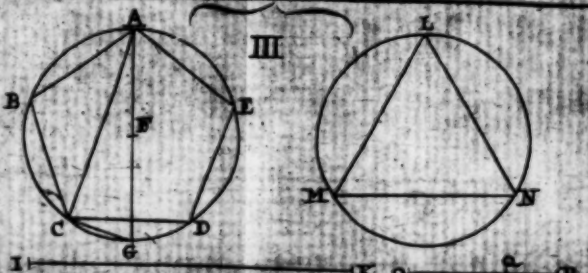

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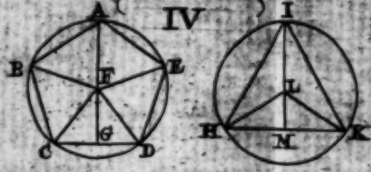

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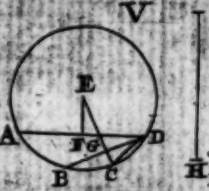

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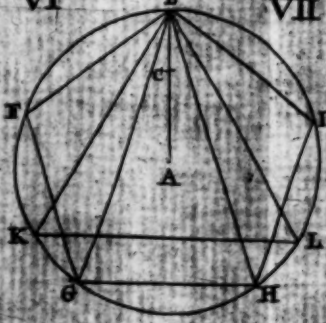
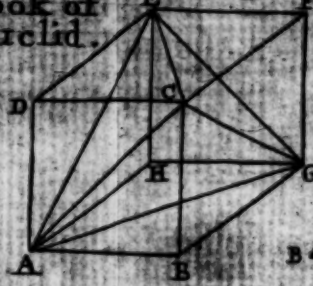


VII

VIII

The Fifteenth Book of Euclid.

I



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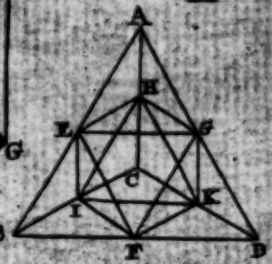

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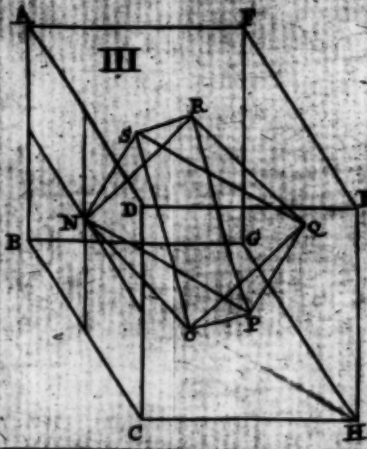

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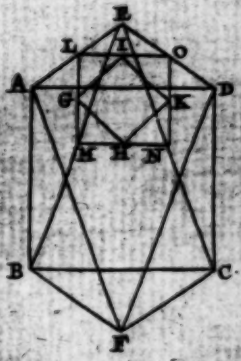

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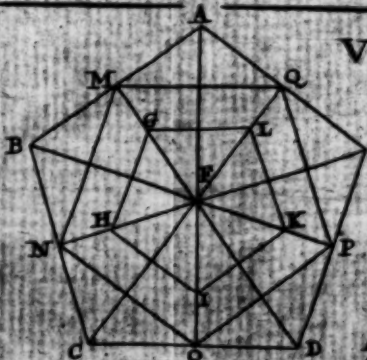

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The: Cross Junior Sculp.

ſort conjoyned with right lines, then will pentagones equal and like to the pentagone GHIKE be deſcribed. Wherefore 12 of ſuch pentagones ſhall conſtitute a dodecaedron; which alſo ſhall be deſcribed in the Icoſaedron, ſeeing the twenty angles of the dodecaedron conſiſt upon the centers of the twenty baſes of the Icoſaedron. Whereby it appears that we have deſcribed a dodecaedron in an Icoſaedron given. Which was to be done.

FINIS.